The mathematics of endo-state spaces in TQFT

3.7.24

Based on [2307.09229] & work in progress

1. Endo-state spaces

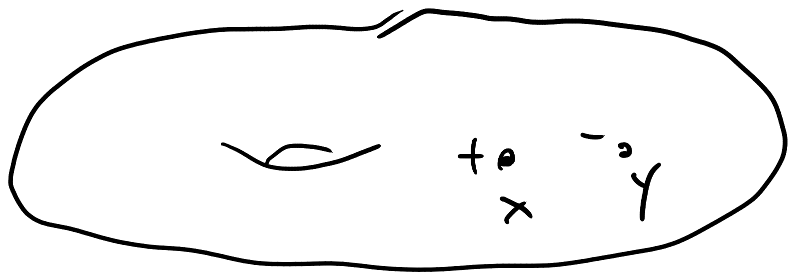
Let $Z: \mathrm{Bord}_3^{or}(\mathcal{C}) \longrightarrow \mathrm{Vect}$ be a TQFT

with "particles" labelled by a TCC $\mathcal{C}$ $(\otimes, *, \mathrm{br})$

Σ

"

$Z\Big($



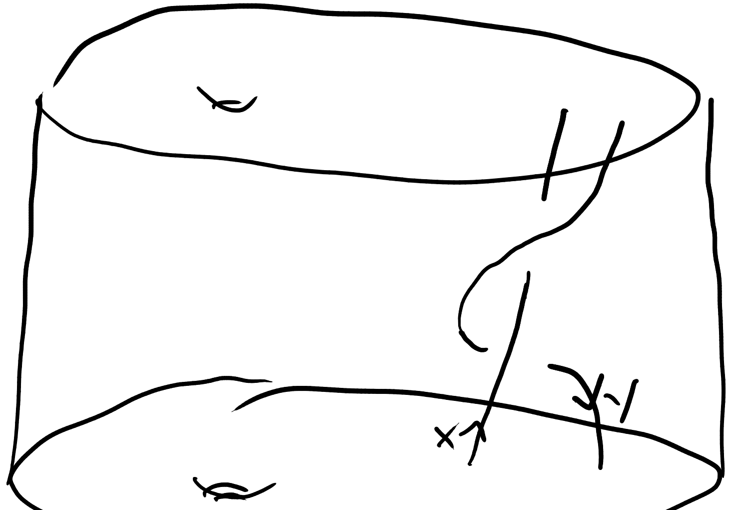
$\Big) \in \mathrm{Vect}$

"state space"

$x \in \mathcal{C}$

$y \in$

$Z\Big($



Σ'

$\Big): Z(\Sigma) \longrightarrow Z(\Sigma')$

Σ

linear map

braiding $x \otimes y \xrightarrow{\simeq} y \otimes x$ gives operation for top quantum computing.

• Construction of Z for a MTC $\mathcal{C}$ from Reshetikhin-Turaev (RT),

$$Z(\Sigma) = \mathrm{Hom}_{\mathcal{C}}(z_\Sigma, x \otimes y^*)$$

$$z_\Sigma \in \mathcal{C}$$

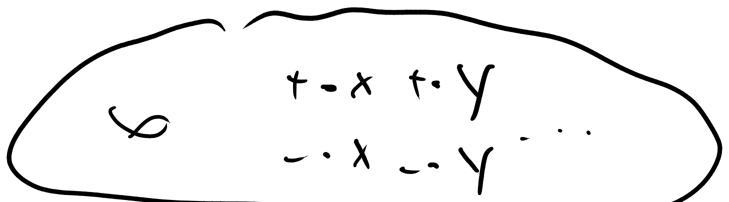
- for $\Sigma = S^2$, $z_\Sigma = \mathbb{1}$ | $\Sigma = T^2$,

$$z_\Sigma = \bigoplus_{i \in I} x_i \otimes x_i^*$$

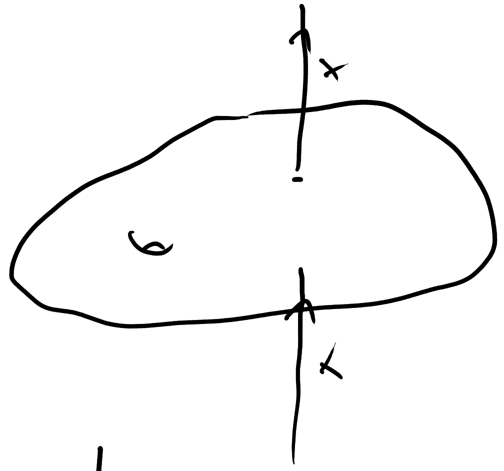
Def An <u>endo-state space</u> is a state space of the type

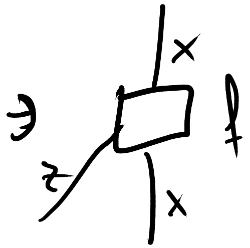
$Z\Big($  $\Big)$,

i.e. x is in-coming & out-going or more generally, with pairing



Physically: „x is preserved"

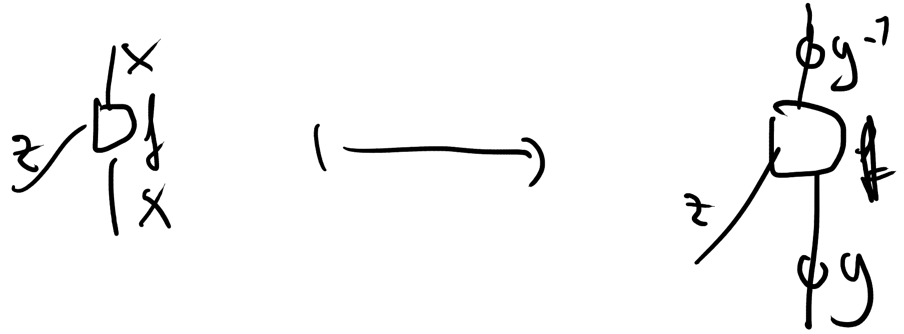


$Z(\Sigma, x) \cong \mathrm{Hom}(Z_\Sigma \otimes x, x) \ni$ 

Two structures occur:

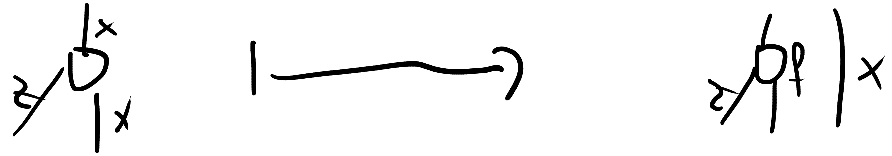
A) Gauge action: $g \in G_x$ gives

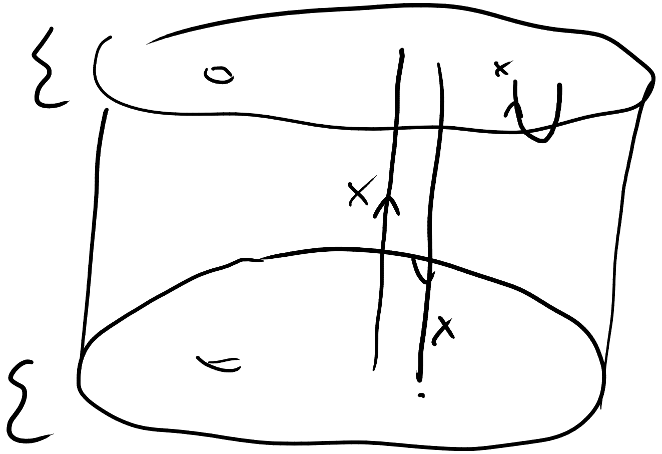
$$Z(\Sigma, x) \xrightarrow{g} Z(\Sigma, x)$$



„change of coordinates"

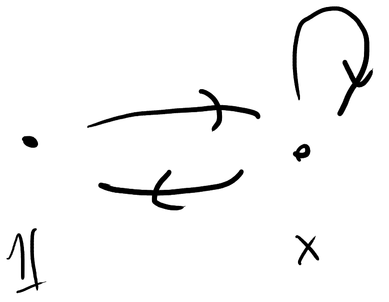
B) stabilization $Z(\Sigma, x) \xrightarrow{i_x} Z(\Sigma, x \otimes x)$



$i_x = Z($ 

2. Quivers

A quiver Q is a finite directed graph:

- Vertices I
- $Q =$ 

(1)

$\left(\mathrm{Hom}(Z(\overset{x}{\ddot{\ }}) \otimes x_i, x_j)\right)_{x_i \in \{\mathbb{1}, x\},\ x_j \in}$

- A Q-module is an assignment
  - $V_i \in \mathrm{Vect}\ \forall i \in I$
  - $f_\alpha : V_i \to V_j$

    $\forall \alpha : i \to j$ edge in Q

- A morphism $h : (V, f) \to (V', f')$ is

  $(h_i : V_i \to V'_i)_{i \in I}$ that commutes with f_α & f'_α

$\rightsquigarrow$ category $\mathrm{mod}\, Q = \mathrm{mod}\, \mathbb{k}Q$

$\uparrow$ path algebra

$\rightsquigarrow$ (ss) simple, indecomposables,
finite rep type $\Leftrightarrow$ Q is a Dynkin quiver

$\rightsquigarrow$ For fixed $(V_i)_{i\in I}$ consider $\mathrm{mod}_V Q \curvearrowleft GL_V$
$ \overset{\shortparallel}{V} \qquad\qquad\qquad\qquad \overset{\shortparallel}{\prod GL V_i}$

action by conjugation $\overset{GIT}{\rightsquigarrow}$ $M_V(Q)$ alg variety that parametrizes ss modules.

Prop Quivers $\overset{1:1}{\longleftrightarrow}$ Endofunctors $F: \mathcal{A} \to \mathcal{A}$ of a finite ss category $\mathcal{A}$

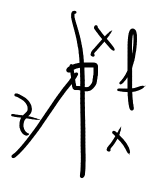
<u>Proof</u> Given I, Q is given by $\mathbb{N} \times \mathbb{N}$-matrix Q_{ij}, same data occurs for such endofunctors.

Example: $\mathcal{A}$ monoidal, $z \in \mathcal{A}$ $\rightsquigarrow$ $Q_z = z \otimes - : \mathcal{A} \to \mathcal{A}$

- $\mathcal{A} = \mathrm{Fib} = \langle 1, x \rangle_{x^2 = 1 \oplus x}$, $\underset{\overset{\shortparallel}{x}}{Q_z} = Q$ from (1)

Fact: · mod $Q \ni (x, f: Qx \to x)$, if $Q = Q_z$,

$Q: \mathcal{A} \to \mathcal{A}$

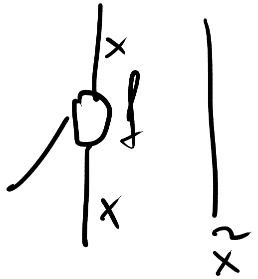


$x \in \mathcal{A}$

· conjugation actions correspond

Additional structures on mod Q

If $z \in \mathcal{A}$ braided , $Q = Q_z$

1) $f \tilde{\otimes} g :=$ 

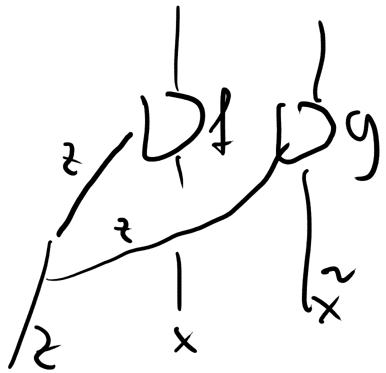
$+$ 

gives

$$\mathrm{mod}_x Q_z \times \mathrm{mod}_{\tilde{x}} Q_z \longrightarrow \mathrm{mod}_{x \otimes \tilde{x}} Q_z$$

monoidal product

2) If $z \in \mathcal{A}$ is an (co) algebra,

$f \otimes g =$ 

$\in \mathrm{mod}_{x \otimes \tilde{x}} Q_z$

Thm [5] If $\mathcal{A}$ is braided fusion, then $\tilde{\otimes}$ gives mod Q_z a rigid monoidal structure

In both cases, the moduli spaces get

$$M_x(Q_z) \times M_y(Q_z) \to M_{x\otimes y}(Q_z)$$

<u>Example</u> With $G \subset SL_2(\mathbb{C})$ finite $\mathcal{A} = \mathrm{Rep}(G)$

with $z = \mathbb{C}^2$ we get doubles of extended Dynkin quivers Q_z

Back to TQFT:

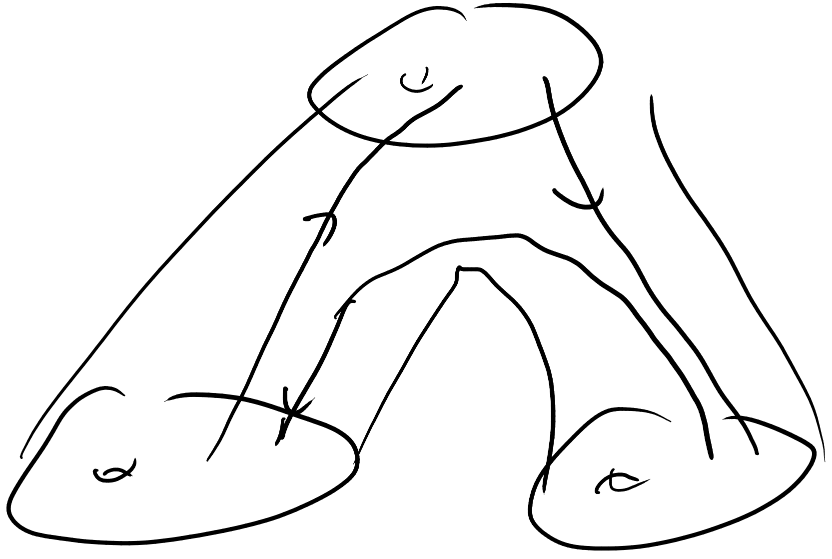
$$\mathcal{Z}(\Sigma_g, x) = \mathrm{Hom}(Z_g \otimes x, x) = \mathrm{mod}_x(Q_{Z_g})$$

$$\downarrow$$

$$M_x$$

3) AF - algebra

$Z(\Sigma_{g,x})$ is also an algebra via



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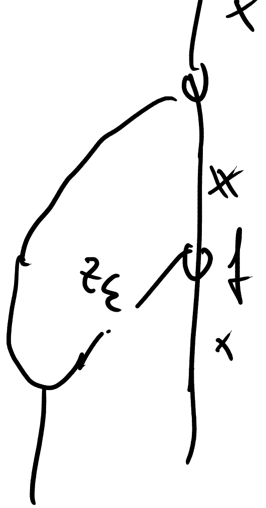
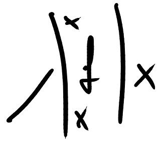


$\|$

${}^g\mathcal{A}^1_x$

$\mathcal{A}^1_x \xrightarrow{\ i_x\ } \mathcal{A}^2_x$ algebra morphism



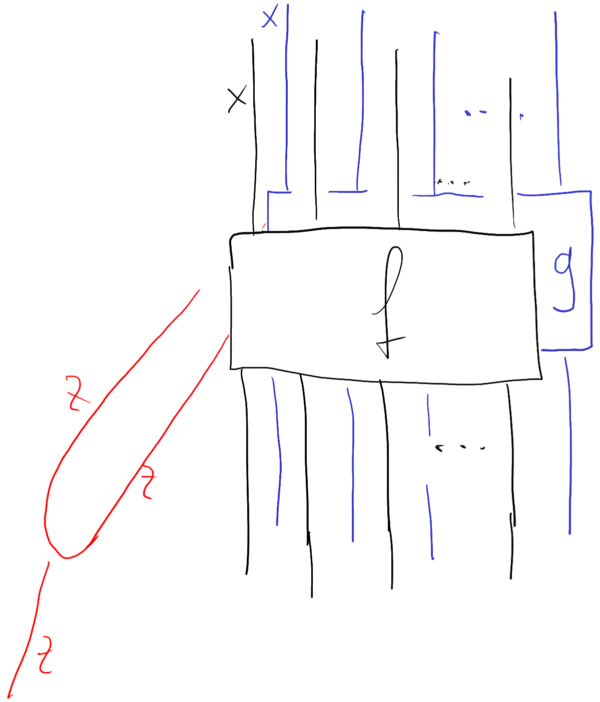
$\mathcal{A}^1_x \xrightarrow{\ i\ } \mathcal{A}^2_x \longrightarrow \cdots \longrightarrow {}^g\mathcal{A}_x$ limit AF C^*-algebra

$\rightsquigarrow K_0(\mathcal{A}_x)$ "is" complete invariant [Elliot thm]

Fusion product:

[Aasmund-Evans] building on [Erlijman-Wenzl] defined $g=0$ case of

$$ {}^g\mathcal{A}^n_x \times {}^g\mathcal{A}^n_x \xrightarrow{\ *\ } {}^g\mathcal{A}^{2n}_x $$

$f * g =$ 

satisfies $if * ig = i^2(f*g)$

Thm [AE, S] : $\cdot$ $*$ defines ${}^g\mathcal{A}_x \times {}^g\mathcal{A}_x \to {}^g\mathcal{A}_x$ associative

$\cdot$ $K_0({}^g\mathcal{A}_x)$ is a ring

Example: Fib $= \mathcal{A}$ $z = x$ we get a diagram for $i = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $K(\mathcal{A}_x)$ computes via

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}} \cdots \to K_0(\mathcal{A})$$

Combine

$$\begin{tikzcd} \mathrm{mod}_{x^n}(\mathbb{Q}_\ell) \arrow[r, "i"] \arrow[d, "\pi"] & \mathrm{mod}_{x^{n+1}}(\mathbb{Q}_\ell) \arrow[d] & & \\ M_x^n \arrow[r, "i"] & M_x^{n+1} \arrow[r] & \cdots \arrow[r] & \hat{M}_x \end{tikzcd}$$

limit variety

gets associative product.

Literatur

- [B. & S. Fusion quiver : 2307.09229]
- [Aaserud-Evans : K-theory of AF-algebras from braided C*-tensor categories 2020]
- [Erlijman-Wenzl: Subfactors from braided C*-tensor categories]
- [Marcolli-Napp : Quantum Computation and Real Multiplication]