Homotopy Theory of Quantum systems

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joint with

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Motivation:

Model independent

Gapped Invertible Phases of Matter

equivalences classes of

Model dependent

Gapped invertible Quantum Systems

Kitaev's Conjectures:

A. Phases are governed by ground state

B. Classifying Spaces

$G_0, G_1, G_2, \ldots$

$\pi_0 G_d$ = Phases in space dim d

$$G_d \simeq \Omega G_{d+1}$$

Get G a loop-spectrum!

2013 Talk @ Simons Center
2015 Talk @ IPAM

E.g. Quantum systems: QFT's

Gapped Invertible Phases of Matter

Deformation classes of

Reflection positive invertible extended field theories

Theorem (Freed-Hopkins, Grady)

For bosonic phases, no internal symmetries

$$G = \Sigma^2 I_{\mathbb{Z}} MSO$$

so $G_0 = K(\mathbb{Z},2)$ $\quad G_1 = K(\mathbb{Z},3)$ $\quad G_2 = \mathbb{Z} \times K(\mathbb{Z},4)$ ...

This talk is not about QFT's ...

... but about lattice models

Motivation:

Model independent

Gapped Invertible Phases of Matter

equivalences classes of

Model dependent

Gapped invertible Quantum systems (q.s.)

Kitaev's Conjectures:

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2013 Talk @ Simons Center
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This talk:

I. Quantum systems as spaces of states on a lattice model

... topological monoid

II Passage from quantum systems Phases as group completion

... output ∞-loop space $\mathcal{Q}_d^\infty$

III (time permitting) Parametrized q.s.

Hope: $G_d \simeq \mathcal{Q}_d^\infty$ for appropriate

Not in this talk:
- specific choices $\mathcal{Q}_d^\infty$
- connection $\mathcal{Q}_d^\infty \overset{?}{\simeq} \Omega \mathcal{Q}_{d+1}^\infty$

I. Quantum Systems as Spaces of States

Hilb : obj finite dimensional non-zero $\mathbb{C}$-Hilbert spaces

mor linear isometries

monoidal product : $\otimes$

Picture $(d=1)$

$d \geq 0$ Spatial dimension.

$\mathbb{Z}^d \subset \mathbb{R}^d$: location of particles (sites)

"lattice"

$\mathcal{H} \in$ Hilb Let $B(\mathcal{H})$ linear operators on $\mathcal{H}$: observables localized at single site.

$$\mathfrak{A}_d(\mathcal{H}) = \overline{\bigotimes_{v \in \mathbb{Z}^d} B(\mathcal{H})}$$

: (C^*)-algebra of observables.

States:

$$\mathfrak{A}_d(\mathcal{H})^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{A}_d(\mathcal{H}),\mathbb{C}) \supset S_d(\mathcal{H}) \supset \mathcal{P}_d(\mathcal{H})$$

linear dual with weak* topology
min. top. s.t. expectation values $A \longmapsto \omega(A)$ are continuous

States
- positive $\omega(A^*A) \geq 0$
- norm one $\omega(I) = 1$

Pure states
extremal points

Energy / time Evolution:

Hamiltonian H "=" $\sum_{\substack{\Lambda \subset \mathbb{Z}^d \\ \text{finite}}} H_\Lambda$ ← self-adjoints on $\bigotimes B(\mathcal{H})$

Locality

Time evolution $A \longmapsto \alpha_t(A) =$ "$e^{itH} A e^{-itH}$"

invertible & gapped $\begin{cases} \text{unique ground state } \omega \\ \quad \omega \circ \alpha_t = \omega \quad + \text{ minimizes energy} \\ \sigma(H) \subset \{E_{min}\} \cup [E_{min} + \Delta, \infty) \quad \Delta > 0 \end{cases}$

Kitaev's Conjectures: (See Xiong arxiv: 1906.02892)

A. Phases are governed by ground state

$$\left\{ \begin{array}{c} \text{Gapped invertible} \\ \text{"H"} \end{array} \right\} \xrightarrow[\substack{\simeq \\ \text{weak equivalence}}]{H \longmapsto \omega} \left\{ \begin{array}{c} \text{Ground states} \\ \text{of such "H"} \end{array} \right\} = \mathcal{Q}_d(\mathcal{H}) \subset \mathcal{P}_d(\mathcal{H})$$

Examples "bosonic, no symetry":

$d=0$ $\quad \mathcal{A}_0(\mathcal{H}) = \mathcal{B}(\mathcal{H}) \quad \mathcal{P}_0(\mathcal{H}) \cong \mathbb{P}(\mathcal{H})$

$$\left\{ \begin{array}{c} \text{self-adjoint } H \in \mathcal{B}(\mathcal{H}) \\ E_{min}\text{. multiplicity } 1 \end{array} \right\} \xrightarrow[\simeq]{} \left\{ \begin{array}{c} \text{Eigenspace} \\ E_{min} \end{array} \right\} = \mathcal{Q}_0(\mathcal{H}) = \mathcal{P}_0(\mathcal{H})$$

UPSHOT $\quad \mathcal{Q}_0(\mathcal{H}) := \mathcal{P}_0(\mathcal{H})$

<u>Warning</u>:

<u>Theorem</u> (Pflaum-Spiegel '23 arxiv: 2402.03605)

If $d>0$, $\mathcal{P}_d(\mathcal{H})$ is weakly contractible.

$\mathcal{P}_d(\mathcal{H})$ too big! Contains many "non-physical" states.

$d>0$ $\qquad \mathcal{Q}_d(\mathcal{H}) \subsetneq \mathcal{P}_d(\mathcal{H})$

Kitaev's Conjectures:

A. Phases are governed by ground state

$$\left\{ \begin{array}{c} \text{Gapped invertible} \\ \text{"}H\text{"} \end{array} \right\} \xrightarrow[\simeq \text{ weak equivalence}]{H \longmapsto \omega} \left\{ \begin{array}{c} \text{Ground states} \\ \text{of such "}H\text{"} \end{array} \right\} = \mathcal{Q}_d(\mathcal{H}) \subset \mathcal{P}_d(\mathcal{H})$$

Examples "bosonic, no symmetry":

$d=1$

$\cdots \overset{B(\mathcal{H})}{\bullet} \; \overset{B(\mathcal{H})}{\bullet} \; \overset{B(\mathcal{H})}{\bullet} \cdots$

$$\left\{ \begin{array}{c} \text{Gapped invertible} \\ \text{"}H\text{"} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Injective Matrix} \\ \text{Product States} \\ \text{with physical} \\ \text{on-site Hilbert} \\ \text{space } \mathcal{H}. \end{array} \right\} = \mathcal{Q}_1(\mathcal{H}) \subsetneq \mathcal{P}_1(\mathcal{H})$$

where $\mathrm{MPS}^{inj}(\mathcal{H}) = \left\{ \text{Injective Matrix Product States with physical on-site Hilbert space } \mathcal{H}. \right\}$

Hastings: arxiv: cond-mat/0701055, 0705.2024

Verstraete-Cirac: arxiv: cond-mat/0505140

Recap: In general $\boxed{\mathcal{Q}_d(\mathcal{H})} \subsetneq \mathcal{P}_d(\mathcal{H}) \Rightarrow$ pure state space

Goal: Identify structure of $\mathcal{Q}_d(\mathcal{H}) \subset \mathcal{P}_d(\mathcal{H})$ coming from physical considerations.

$\psi_v, v \in \mathbb{Z}^d$

$\bigotimes \psi_v \in \bigotimes_{\mathbb{Z}^d} \mathcal{H}$

$$\mathfrak{A}_d(\mathcal{H}) = \overline{\underset{\substack{\Lambda \subseteq \mathbb{Z}^d \\ \text{finite}}}{\mathrm{colim}} \bigotimes_{\Lambda} B(\mathcal{H})}$$

$\mathfrak{A}_d(\mathcal{H})_{loc}$

$$\underset{\Lambda}{A} \longmapsto \underset{\Gamma}{I \otimes A \otimes I}$$

Properties of States:

Functoriality: $f : \mathcal{H} \to \mathcal{K}$ a lin. isometry

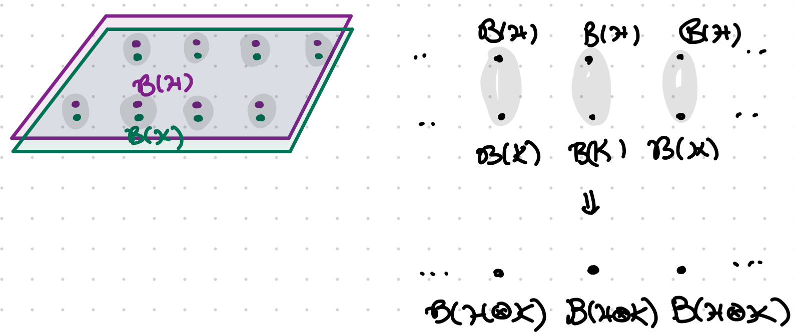
closed embedding

$$\mathcal{P}_d(f) : \mathcal{P}_d(\mathcal{H}) \xrightarrow{\subseteq} \mathcal{P}_d(\mathcal{K})$$

$$\omega \longmapsto \omega \circ \mathrm{Ad}\left(\bigotimes_{d} f^*\right)$$

for $\mathrm{Ad}(f^*)(A) = f^* A f$

Stacking:



$$\mathcal{P}_d(\mathcal{H}) \times \mathcal{P}_d(\mathcal{K}) \longrightarrow \mathcal{P}_d(\mathcal{H} \otimes \mathcal{K})$$

$$(\omega_{\mathcal{H}}, \omega_{\mathcal{K}}) \longmapsto \omega_{\mathcal{H}} \otimes \omega_{\mathcal{K}}$$

<u>Theorem</u> (BHMPQS; arxiv: 2303.07431)

$\mathcal{P}_d : \mathrm{Hilb}^{\otimes} \longrightarrow \mathrm{Top}^{\times}$ is a topologically enriched lax monoidal functor.

<u>Remark</u>: Quantum Systems also should form this structure.

<u>Definition</u> A quantum state type $\mathcal{Q}_d$ is a sub (lax monoidal) functor of $\mathcal{P}_d$

s.t. • $\mathcal{Q}_d(f)$ are closed embeddings

• $\mathcal{Q}_d(\mathcal{H}) \subset \mathcal{P}_d(\mathcal{H})$ contains the trivial states.

$\mathcal{Q}_d$ is invertible if for each $\omega \in \mathcal{Q}_d(\mathcal{H})$, there is $\omega' \in \mathcal{Q}_d(\mathcal{H}')$ s.t.

$\omega \otimes \omega' \in \mathcal{Q}_d(\mathcal{H} \otimes \mathcal{H}')$ is in the component of the trivial states.

Triviality : $\Psi_v \in \mathbb{P}(\mathcal{H}), \ v \in \mathbb{Z}^d$

$$\mathcal{P}_d(\mathcal{H}) \ni \bigotimes_{v\in\mathbb{Z}^d} \Psi_v : \bigotimes_{v\in\mathbb{Z}^d} A_v \longmapsto \prod_v \langle \Psi_v, A_v \Psi_v \rangle$$

"Trivial State"

If $\Psi \in \mathbb{P}(\mathcal{H})$ is the vacuum,

$$\Psi_v \xrightarrow{\text{path}} \Psi \quad \text{in } \mathbb{P}(\mathcal{H})$$

$\therefore$ trivial states can be deformed to vacuum $\boxed{\Psi = \bigotimes_{v\in\mathbb{Z}^d} \Psi \in \mathcal{P}_d(\mathcal{H})}$

Note: all states for $d=0$ are Trivial.

II From Quantum Systems to Phases

A phase is an equivalence class of states ω where $\omega \equiv \omega'$ is generated by:

1) Deformation : $\omega \equiv \omega'$ if there is a path in $\mathcal{Q}_d(\mathcal{H})$ $\omega \to \omega'$

2) Stabilization : $\omega \equiv \omega'$ if there are trivial states t, t' s.t. $\omega \otimes t = \omega' \otimes t'$.

think reduced K-theory

Group completion... Fix $\mathcal{H} \in \mathrm{Hilb}$, $\dim \mathcal{H} \geq 2$

$$\mathcal{Q}_d^{\otimes} = \coprod_{n \geq 0} \mathcal{Q}_d(\mathcal{H}^{\otimes n})$$ Space of quantum systems. (top. monoid)

$$\mathcal{Q}_d^{\infty} = \operatorname*{colim}_{\otimes \psi} \mathcal{Q}_d(\mathcal{H}^{\otimes n})$$ classifying space for phases

$\mathcal{Q}_d : \mathrm{Hilb}^{\otimes} \longrightarrow \mathrm{Top}^{\times}$

$\mathcal{Q}_d : \mathrm{Vect}^{\oplus} \longrightarrow \mathrm{Top}$

equivalence relation implemented by 1) Taking π_0
2) the colimit.

May's Recognition Principle: If Q_d is an invertible quantum state type Q_d^∞ is a grouplike E_∞-space. So Q_d^∞ is an infinite loop space

$$\Omega B Q_d^{\otimes} \simeq \mathbb{Z} \times Q_d^\infty .$$

Examples "bosonic, no symetry":   Kitaev's spectrum $G = (G_0, G_1, \dots)$

d=0

$$G_0 = Q_0^\infty = \operatorname*{colim}_{\otimes \psi} \mathbb{P}(\mathcal{H}^{\otimes n}) \simeq \mathbb{CP}^\infty \simeq K(\mathbb{Z},2).$$

d=1   Pre-Theorem (Beaudry-Hermele-Pflaum-Qi-Spiegel-Stephen)
to appear

$$G_1 = Q_1^\infty \simeq \operatorname*{colim}_{\otimes \psi} MPS^{inj}(\mathcal{H}^{\otimes n}) \simeq K(\mathbb{Z},3)$$

$\bigcup_n$ " BPU(n)

III Parametrized systems.

Parameter space : X a finite CW complex.

$$Q_d^{\otimes} = \bigsqcup_n Q_d(\mathcal{H}^{\otimes n})$$

Definition: A parametrized q.s. is

$$\omega \in \mathrm{Map}(X, Q_d^{\otimes}) =: Q^d(X)$$

We say ω is trivial if it is constant at a trivial state.

Parametrized Phases: $\omega \equiv \omega'$
1) $\omega \simeq \omega'$
2) $\exists\, t, t'$ trivial s.t. $\omega \otimes t = \omega \otimes t'$.

Phases parametrized by X:

$$PQ^d(X) := [X, Q_d^{\infty}] \cong K_0(\pi_0 Q^d(X)) \times [X, \mathbb{Z}]$$

Example

d=0

$G^0(X) = PQ^0(X) = \mathrm{Line}_{\mathbb{C}}(X)$ corresponding to ground state line bundle... "Berry phase"

for Kitaev's cohomology thy G

$G_0 = K(\mathbb{Z},2)$

canonical example ...

$$X = S^2 \longrightarrow M_2(\mathbb{C})$$

$$\vec{x} \longmapsto \vec{x}\cdot\vec{\sigma} = x_1\sigma^1 + x_2\sigma^2 + x_3\sigma^3$$

H

Phase invariant : $c_1 \neq 0 \in H^2(S^2;\mathbb{Z})$

Example

$$G^!(X) = PQ^!(X) = \mathrm{Gerbes}_{\mathbb{C}}(X) = H^3(X;\mathbb{Z})$$

$d=1$

arxiv : 2112.07748

arxiv : 2305.07700

Canonical example $X = S^3$

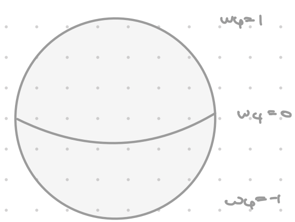
$$(\omega_1,\omega_2,\omega_3,\omega_4) \longmapsto \sum_{p\in\mathbb{Z}} (-1)^p(\omega_1\sigma^1_p + \omega_2\sigma^2_p + \omega_3\sigma^3_p)$$

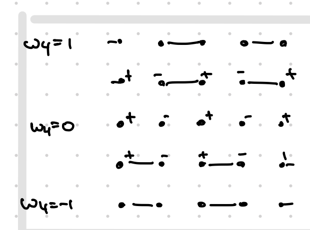
$$+\ \phi^+(\omega) \sum_{p\in 2\mathbb{Z}+1} \sigma^1_p\sigma^1_{p+1} + \sigma^2_p\sigma^2_{p+1} + \sigma^3_p\sigma^3_{p+1}$$

$$+\ \phi^-(\omega) \sum_{p\in 2\mathbb{Z}} \sigma^1_p\sigma^1_{p+1} + \sigma^2_p\sigma^2_{p+1} + \sigma^3_p\sigma^3_{p+1}$$

$$\phi^+(\omega) = \begin{cases} \omega_4 & 0\le\omega_4\le 1 \\ 0 & \text{o/w} \end{cases}$$

$$\phi^-(\omega) = \begin{cases} -\omega_4 & -1\le\omega_4< 0 \\ 0 & \text{o/w} \end{cases}$$





Other work Kapustin, Spodyneiko : arxiv: 2001.03454

Ohyama, Terashima, Shiozaki - arxiv: 2303.04252

Ohyama, Shinsei Ryu arxiv: 2405.05327.

Shiozaki, Heinsdorf, Ohyama: arxiv: 2305.08109

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Thank you.