Singularity Theorems in General Relativity

Context, proof and relevance

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Lorentzian manifolds

A Lorentzian manifold is a pair (M, g), where M is an n-dimensional smooth manifold and g is a metric of signature $(- \ + \ \ldots +)$.

Just as the simplest Riemannian metric is the Euclidean metric,

$$g = dx^1 \otimes dx^1 + \ldots + dx^n \otimes dx^n$$

the simplest Lorentzian metric is the Minkowski metric,

$$g = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + \ldots + dx^n \otimes dx^n$$

Lorentzian manifolds

The **Levi-Civita theorem is valid**: There exists a unique symmetric affine connection compatible with the metric.

Lorentzian metrics give rise to negative inner products between vectors in T_pM. In fact, if

- $g(X,X) < 0$, X is a **timelike** vector
- $g(X,X) > 0$, X is a **spacelike** vector
- $g(X,X) = 0$, X is a **null** vector

Lorentzian manifolds

Metric does not provide a distance function

Lorentzian manifolds are endowed with geodesics but these do not minimize length

The arclength of a curve is called **proper time** and is defined as the time as measured by a clock moving with that geodesic

Timelike geodesics maximize length under certain conditions

General Relativity

Gravitational field is modelled by the curvature of a Lorentzian manifold

Free-falling point particles follow timelike geodesics

Light rays follow null geodesics

Einstein's equations are $R - \frac{1}{2} g\, trR = T$, where R is the Ricci curvature, $\mathrm{tr}R$ the Ricci scalar and T the energy-momentum tensor.

Energy conditions

General conditions imposed on the energy content of the Universe in order to allow only for physically reasonable stress-energy tensors. For example:

Strong energy condition

$$R(X,X) \geq 0, \forall \text{ timelike } X.$$

Null energy condition

$$T(X,X) \geq 0, \forall \text{ null } X.$$

Solutions to Einstein's equations

Minkowski spacetime – vacuum solution without gravitation. Lorentzian analogue of Euclidian spacetime.

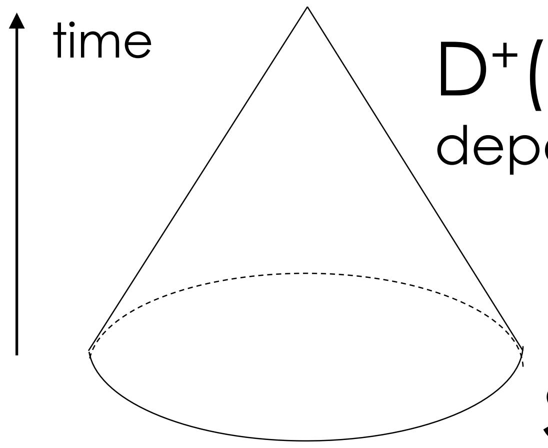
Schwarzschild solution – Spherically symmetric vacuum solution. The metric is of the form,

$$g = -\left(1 - \left(\frac{r_s}{r}\right)^{n-2}\right) dt \otimes dt + \left(1 - \left(\frac{r_s}{r}\right)^{n-2}\right)^{-1} dr \otimes dr + r^2 h$$

- Interior solution has r as a time coordinate
- Interior and exterior can be glued through a horizon at $r = r_s$
- In the interior, timelike geodesics are incomplete, **there is a singularity at $\boldsymbol{r = 0}$**

Globally hyperbolic spacetimes

A spacetime is globally hyperbolic if it admits a surface that is intersected by all inextensible non-spacelike curves exactly once (Cauchy surface).



$D^+(S)$ Future domain of dependence

S Cauchy Hypersurface

$D^+(S)$ is the set of points from which any past-directed inextendible non-spacelike curve will always intersect S.

$D^+(S) \cup D^-(S) = M \Rightarrow$ S is a Cauchy hypersurface

Hawking’s singularity theorem

Raychaudhuri equation

A **congruence** of curves is a family of integral curves of a unit vector field X.

The expansion of a congruence satisfies the Raychaudhuri equation,

$$X \cdot \theta = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}X^{\mu}X^{\nu}.$$

where $\theta = g^{\mu\nu}\nabla_\mu X_\nu$ is the **expansion**, $\sigma_{\mu\nu} = \nabla_{(\mu}X_{\nu)} + \frac{1}{3}\theta h_{\mu\nu}$ is the **shear**, $\omega_{\mu\nu} = \nabla_{[\mu}X_{\nu]}$ is the **vorticity** and $h_{\mu\nu}$ is the spatial metric.

Raychaudhuri equation

If we require the strong energy condition to hold and X to be a timelike geodesic unit vector with vanishing vorticity,

$$\dot{\theta} = -\frac{\theta^2}{3} - 2\sigma^2 - R(X,X) \leq 0.$$

Also,

$$\dot{\theta} \leq -\frac{\theta^2}{3} \quad \Rightarrow \quad \frac{1}{\theta} \geq \frac{1}{\theta_0} + \frac{1}{3}(t - t_0).$$

So the expansion blows up at most at proper time $t = t_0 + \frac{3}{|\theta_0|}$.

Hawking's singularity theorem

Let (M,g) be a globally hyperbolic spacetime satisfying the strong energy condition.

Suppose the expansion of the congruence of future-pointing timelike geodesics orthogonal to a Cauchy hypersurface has negative initial value.

Then (M,g) is singular.

Outline of the proof

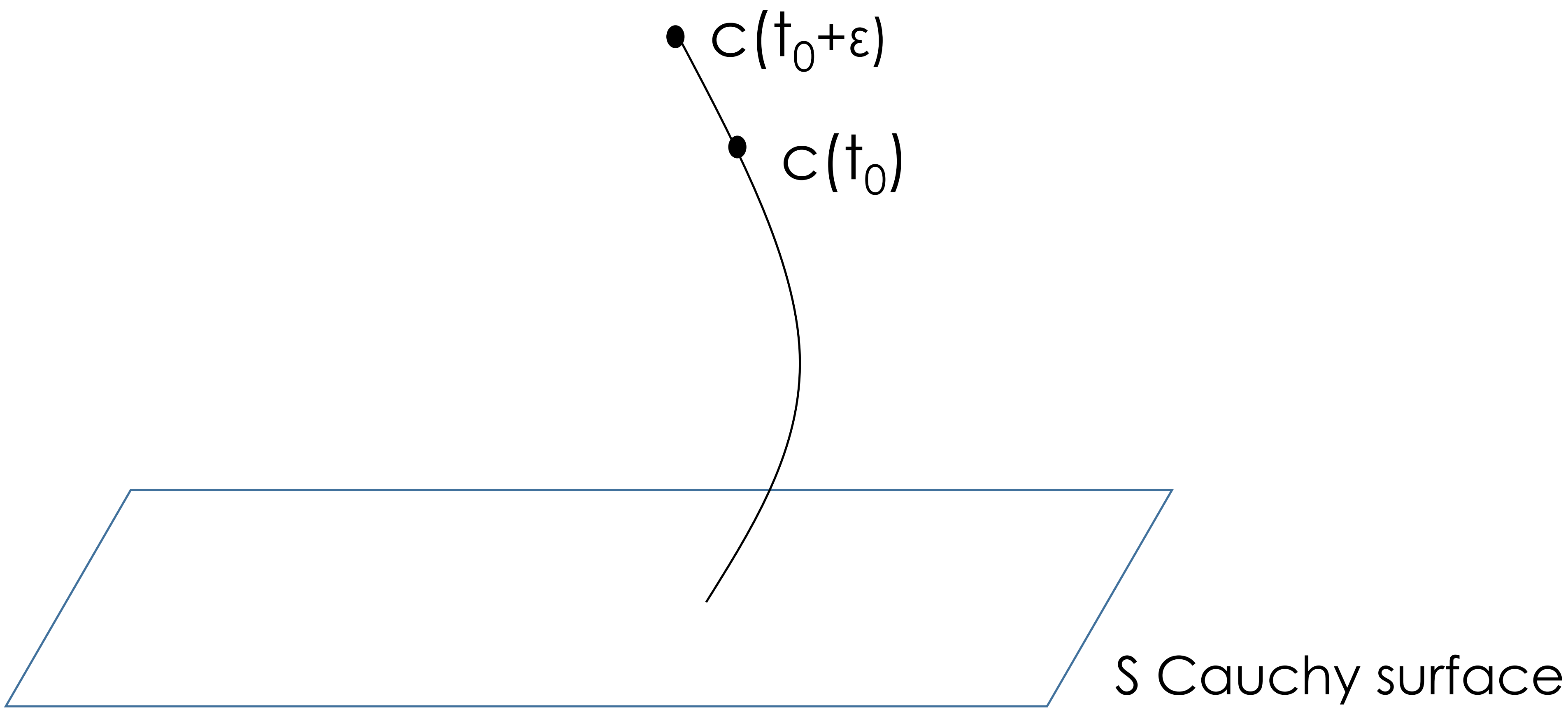
We want to show that no future-directed timelike geodesics orthogonal to S can be extended past proper time $t_0 = \frac{3}{|\theta_0|}$.

- If this were not true, there would be a geodesic c: [0, $t_0 + \varepsilon$] → M
- There would be a timelike geodesic γ maximizing the length between c(0) and c($t_0 + \varepsilon$).
- Past t_0, γ ceases to be maximal.
- We arrive at a contradiction.

Outline of the proof

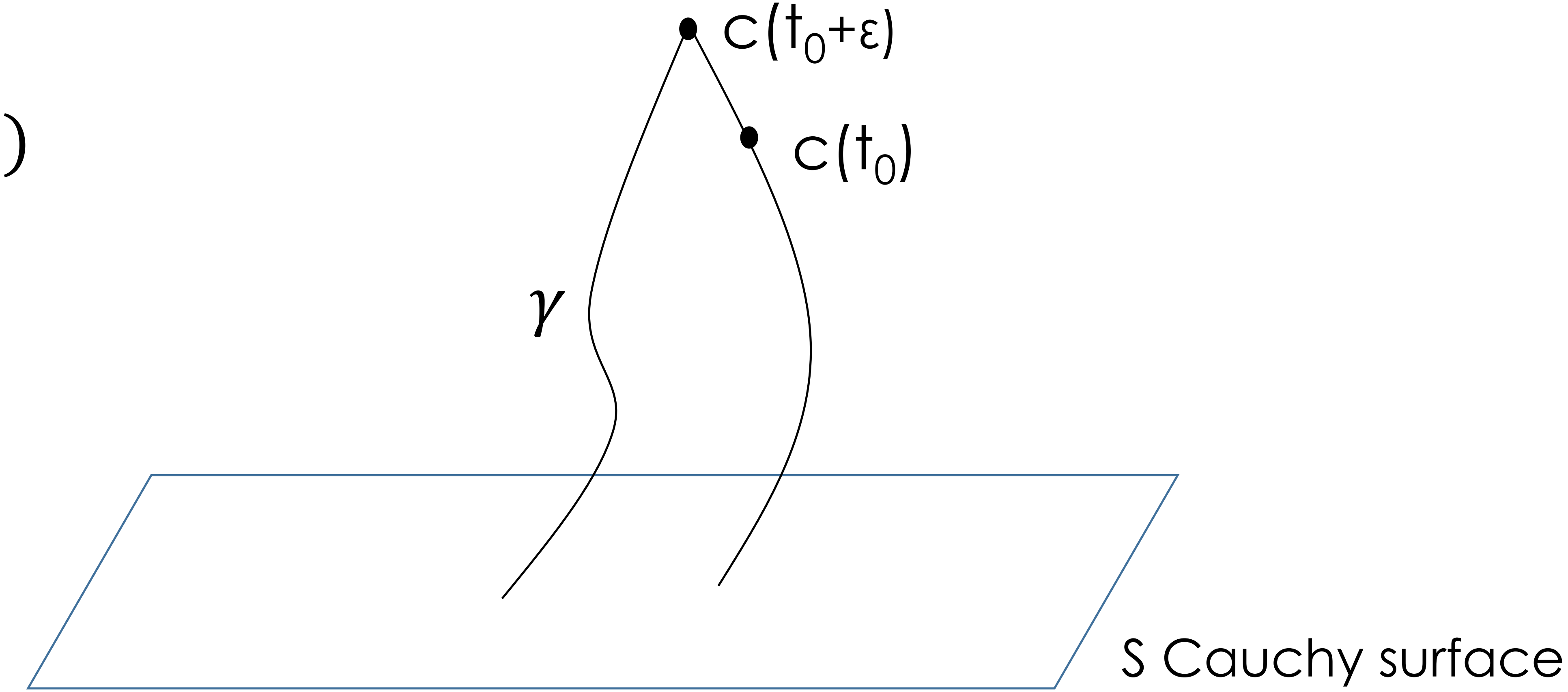
S Cauchy surface

Outline of the proof



Outline of the proof

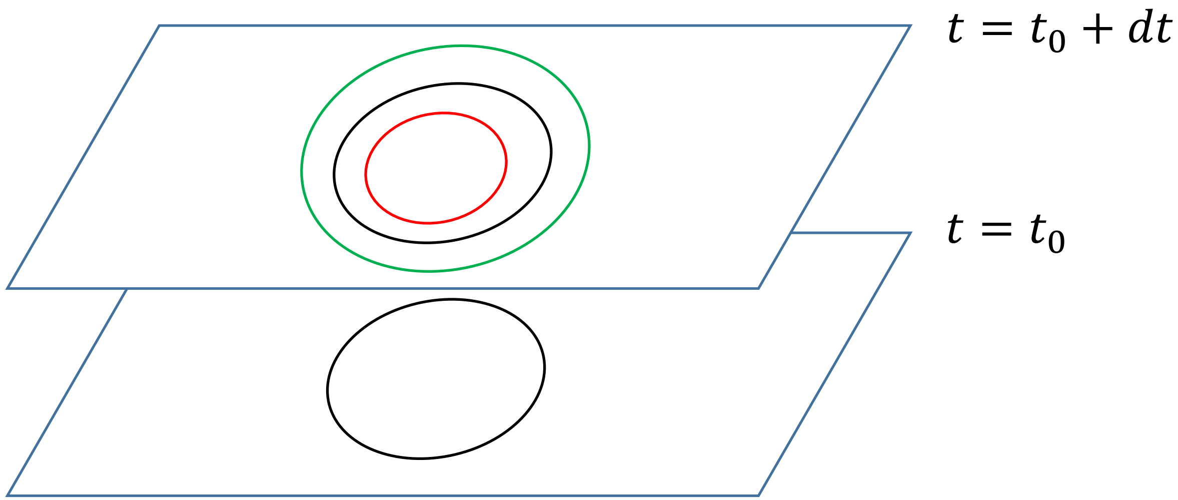
$t(\gamma) > t(c)$



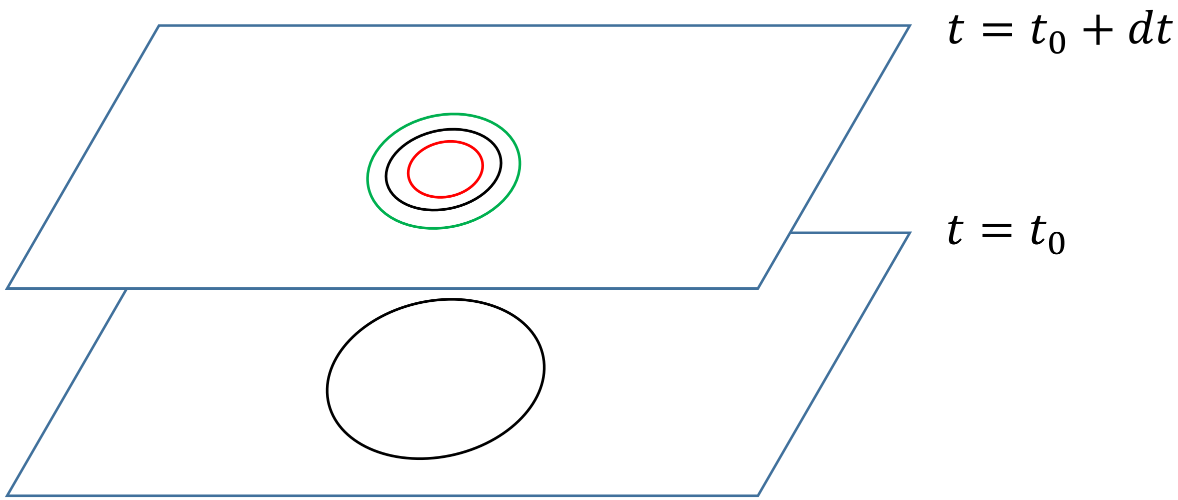
Penrose's singularity theorem

Trapped surfaces

A trapped surface is a 2d embedded submanifold, compact without boundary, such that ingoing and outgoing light rays emerging orthogonally towards the future eventually converge.



Stationary spacetime



Contracting spacetime

Penrose's singularity theorem

Let (M,g) be a connected globally hyperbolic spacetime with a non-compact Cauchy hypersurface S, satisfying the null energy condition.

Suppose S contains a trapped surface.

Then (M,g) is singular.

The first impact

Hawking, Ellis and Geroch took Penrose's theorem in its "past" version with a positive expansion. They proposed that, if GR holds, an initial singularity is unavoidable.

This, together with the discovery of Cosmic Background Radiation, provides strong indications corroborating the Big Bang model.

The relevance at the time

Schwarzschild solution and all the other main exact solutions of GR had singularities

This was thought to be due to the unrealistic degree of symmetry in these simplistic conditions.

The singularity theorems showed otherwise. They proved that under very general conditions, a manifold will unavoidably become singular.