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Dijkgraaf-Witten TQFT and quantum double models with defects

j/w João Faria Martins, Leeds, in progress

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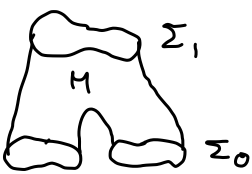
aim

- geometric and gauge theoretical description of defect TQFT
- suitable for computations
- application: defects in quantum double model (= 2d part of 3d DW theory)

1 Dijkgraaf-Witten theory (untwisted)

finite group G $\rightsquigarrow$ symm mon functor $\mathcal{Z}: \mathrm{Cob}_3 \to \mathrm{Vect}_{\mathbb{C}}$



$\Sigma_0 \xrightarrow{\iota_0} M \xleftarrow{\iota_1} \Sigma_1$ $\overset{\mathcal{Z}}{\rightsquigarrow}$ $\mathcal{Z}(M): \mathcal{Z}(\Sigma_0) \to \mathcal{Z}(\Sigma_1)$

cobordism $\qquad$ linear map

$$\mathcal{Z}(\Sigma_j) = \langle \pi_0(\mathrm{GRPD}(\pi_1(\Sigma_j), G)) \rangle_{\mathbb{C}} = \langle \{A: \pi_1(\Sigma_j) \to G\}/\cong \rangle_{\mathbb{C}}$$

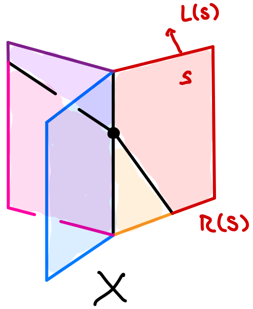
$$\langle [A_0] \,|\, \mathcal{Z}(M) \,|\, [A_1] \rangle = |\{A: \pi_1(M) \to G \,|\, A \circ \pi_1(\iota_j) = A_j\}/\cong\}|$$

$\rightsquigarrow$ compute e.g. with triangulations

② defects

~ lower-dim submanifolds with higher cat data

Stratification



X $n \leq 3$-mfold, closed

$\emptyset = X_{-1} \subseteq X_0 \subseteq \dots \subseteq X_n = X$

$\pi_0(X_k \setminus X_{k-1}) = \{k\text{-strata}\}$ ← k-mfold, $\partial \subseteq \partial X$

homogeneous
neighbourhoods of strata constant

oriented
$(n-1)$-strata have normals

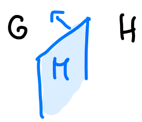
assignment strata ⇝ defect data (untwisted)

codim 0 – finite groups

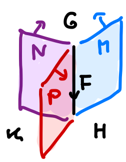
codim 1 – finite sets with group actions

codim 2 – reps of action grpds

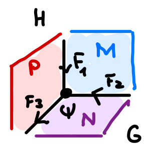
codim 3 – intertwiners of action grpd reps



$G \times H^{op} \circlearrowright M$



$F: M \times N \times P /\!/ G \times H \times K \to \mathrm{Vect}_{\mathbb{C}}$



$\Psi: F_1\pi_1 \otimes F_2\pi_2 \Rightarrow F_3\pi_3 : M \times N \times P /\!/ G \times H \to \mathrm{Vect}_{\mathbb{C}}$

⇒ **defect cobordism category**

$$\mathrm{Cob}_3^{def} = \begin{cases} \text{stratified surfaces + defect data} \\ \text{stratified cobordisms + defect data} \end{cases}$$

defect TQFT symm. monoidal functor $Z: \mathrm{Cob}_3^{def} \to \mathrm{Vect}_{\mathbb{C}}$

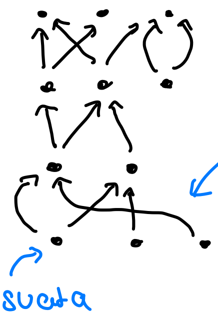
stratification X ⇝ graded graph Q^X ⇝ category $\mathcal{Q}^X$

deg = 3

deg = 2

deg = 1

deg = 0



3 - strata

2 - strata

1 - strata

0 - strata

local incidence

Codim ≤ 1 defect data ⇝ functor $D: \mathcal{Q}^X \to \mathrm{Grpd}$

codim 0 - stratum $t \to \bullet /\!/ G_t$

codim 1 - stratum $s \to M /\!/ G_{L(s)} \times G_{R(s)}$

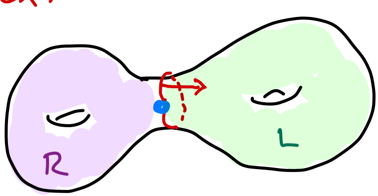
$\Big\}\ D_{cd\le 1}: \mathcal{Q}^X_{cd\le 1} \to \mathrm{Grpd}$

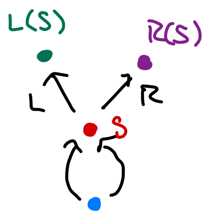
$\Rightarrow D = \mathrm{Ran}_I D_{cd\le 1}: \mathcal{Q}^X \to \mathrm{Grpd}$ $\quad I: \mathcal{Q}^X_{cd\le 1} \to \mathcal{Q}^X$ inclusion

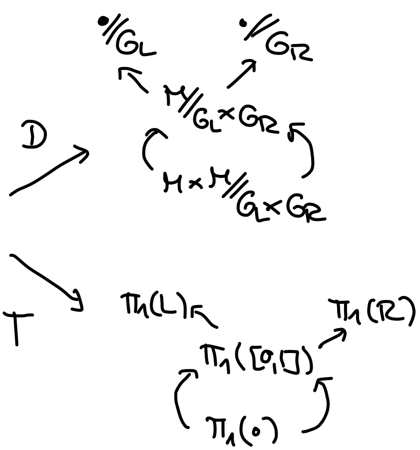
topological data ⇝ functor $T: \mathcal{Q}^X \to \mathrm{Grpd}$

$s \xrightarrow{q} t \quad \overset{T}{\rightsquigarrow} \quad \pi_1(\hat{s}) \xrightarrow{\pi_1(c)} \pi_1(\hat{t})$

Ex:







③ Construction of defect TQFT

defect data for codim ≤ 1 - strata

Def gauge groupoid of stratified $n\leq 3$-mfold X

$$\mathcal{A}/\!/\mathcal{G} = \int_{v\in Q^X} \mathrm{GRPD}(T(v), D(v))$$

⇝ unpack:

interval grpd $\bullet \longrightarrow \bullet$

gauge field nat. transf $\mathcal{A} \ni A : T \Rightarrow D$

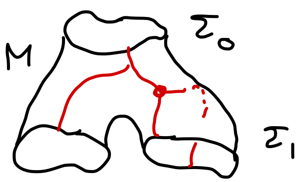
gauge transformation nat. transf. $\mathcal{G} \ni \gamma : T\times I \Rightarrow D$

$$\begin{tikzcd} T \arrow[r, "\iota_0"] \arrow[dr, "A_0"'] & T\times I \arrow[d, "\gamma"] & T \arrow[l, "\iota_1"'] \arrow[dl, "A_1"] \\ & D & \end{tikzcd}$$

defect data for codim ≥ 2 - strata

⇝ diagram for symm monoidal Category $\mathrm{Vect}_{\mathbb{C}}^{\mathcal{A}/\!/\mathcal{G}}$

⇝ evaluation: morphism in $\mathrm{Vect}_{\mathbb{C}}^{\mathcal{A}/\!/\mathcal{G}}$



$$\xrightarrow{\quad G \quad}$$

$$\begin{tikzcd} F_0 : \mathcal{A}^0/\!/\mathcal{G}^0 \arrow[r] & \mathrm{Vect}_{\mathbb{C}} \\ \mathcal{A}^M/\!/\mathcal{G}^M \arrow[u, "P_0"] \arrow[d, "P_1"'] & G' : F_0 P_0 \Rightarrow F_1 P_1 \\ F_1 : \mathcal{A}^1/\!/\mathcal{G}^1 \arrow[r] & \mathrm{Vect}_{\mathbb{C}} \end{tikzcd}$$

$\langle P_0, P_1\rangle : \mathcal{A}^M/\!/\mathcal{G}^M \to \mathcal{A}^0/\!/\mathcal{G}^0 \times \mathcal{A}^1/\!/\mathcal{G}^1$ fibration

Thm [JFM, CM]

1. ⇒ Symm monoidal functor $G : \mathrm{Cob}_3^{def} \to \mathrm{Span}(\mathrm{RepGrpd})$
2. ⇒ Symm monoidal functor $L : \mathrm{Span}(\mathrm{RepGrpd}) \to \mathrm{Vect}_{\mathbb{C}}$

⇒ defect TQFT $Z : \mathrm{Cob}_3^{def} \to \mathrm{Vect}_{\mathbb{C}}$

④ Examples: quantum double models

Z: surface with defects $\to$ vector space $Z(\Sigma)$

no defects: $Z^G_{DW}(\Sigma)$ = ground state of quantum double model for $\mathbb{C}[G]$

$\rightsquigarrow$ generalisation: „ground" state with defects

dual to stratification

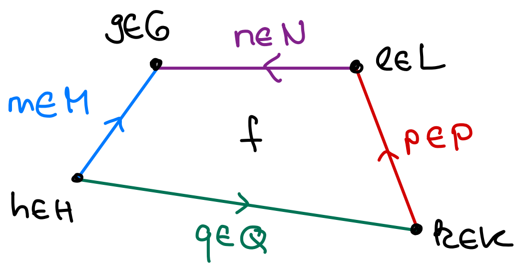
Ex 1 graph $\Gamma \subset \Sigma$, $\Sigma \setminus \Gamma = \coprod$ Discs

defect data
$v \in V_\Gamma \rightsquigarrow$ finite group G_v
$e \in E_\Gamma \rightsquigarrow$ finite $G_{t(e)} \times G^{op}_{s(e)}$ set M_e
$f \in F_\Gamma \rightsquigarrow$ rep $\rho_f : \mathcal{G}_f \to \mathrm{Vect}_\mathbb{C}$

gauge groupoid $\quad \mathcal{A}/\!\!/\mathcal{G} \cong \prod_{e \in E_\Gamma} M_e /\!\!/ \prod_{v \in V_\Gamma} G_v$

functor $\quad \rho = \bigotimes_{f \in F_\Gamma} \rho_f \pi_f : \mathcal{A}/\!\!/\mathcal{G} \to \mathrm{Vect}_\mathbb{C}$

generalised ground state $\quad Z(\Sigma) = \lim \rho$



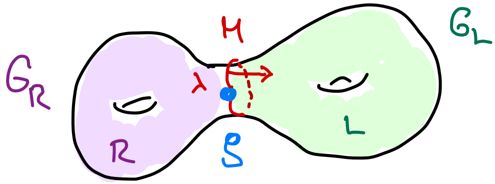
if $G_v = G \;\; \forall v \in V$
$M_e = G \;\; \forall e \in E$

$$\rho_f(m,n,p,q) = \begin{cases} \mathbb{C} & m^{-1}npq = 1 \\ 0 & \text{else} \end{cases}$$

$\Rightarrow$ ground state of quantum double model

$(g,h,k,\ell) \triangleright m = g \triangleright m \triangleleft h^{-1}$

Ex 2



gauge fields $\quad \mathcal{A} \ni A = \begin{cases} \text{functors} \\ A_L : \pi_1(L) \to G_L \quad A_R : \pi_1(R) \to G_R \\ m \in M \end{cases}$

gauge transformations $\quad \mathcal{G} \ni \gamma = (g_L, g_R) \in G_L \times G_R$

$$A_L \to g_L A_L g_L^{-1}$$
$$A_R \to g_R A_R g_R^{-1}$$
$$m \to g_L \triangleright m \triangleleft g_R^{-1}$$

defect vertex $\quad \rho : M \times M /\!/_{G_L \times G_R} \to \mathrm{Vect}_{\mathbb{C}}$

given by orbit $\mathcal{O}$ + rep σ: $\mathrm{Stab}_{\mathcal{O}} \to \mathrm{Aut}_{\mathbb{C}}(V)$

generalised ground state

$$\mathcal{Z}(\Sigma) = V \otimes_{\mathbb{C}[G_L \times G_R]} \langle \mathcal{A}^* \rangle_{\mathbb{C}}$$

$\mathcal{A}^*$: triples (A_L, A_R, m)

$A_L(\lambda) \triangleright m \triangleleft A_R(\lambda)^{-1} = m$

$G_L \times G_R$ - action induced by σ

- $\rho(m, m') = \begin{cases} \mathbb{C} & m = m' \\ 0 & \text{else} \end{cases} \quad \rightsquigarrow \quad \mathcal{Z}(\Sigma) = \langle \mathcal{A}^* / \sim \rangle_{\mathbb{C}}$

- $M = G_L = G_R = G \quad \rightsquigarrow \quad \mathcal{Z}(\Sigma) = \langle \pi_0(\mathrm{GRPD}(\pi_1(\Sigma), G)) \rangle_{\mathbb{C}}$

  $\rho(g, g') = \begin{cases} \mathbb{C} & g = g' \\ 0 & \text{else} \end{cases} \qquad = \mathcal{Z}^G_{DW}$