Geometria em Lisboa seminar

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A spherical Logarithm map

(joint with V. Batyrev, M. Harada, J. Hofscheier)

Review of amoebas & trop. geo. :

$T = \mathbb{R}^n \times (S^1)^n$

$T = (\mathbb{C}^*)^n \qquad T_K = (S^1)^n \qquad \mathrm{Log}_t : T \longrightarrow \mathbb{R}^n = N_{\mathbb{R}}$

$\mathrm{Log}_t(x_1, \ldots, x_n) = (\log_t |x_1|, \ldots, \log_t |x_n|)$

$Y \subset T$ subvariety e.g. hypersurface

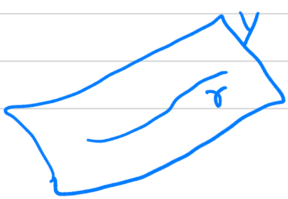
$\underbrace{\mathrm{Log}_t(Y)}_{\text{amoeba of } Y} \longrightarrow \mathrm{trop}(Y) =$ supp. of a polyhedral fan

[GKZ]

Recall: $K = \mathbb{C}\{\{t\}\}$ field of Puiseux series $= \overline{\mathbb{C}((t))}$

$\mathrm{val} : (K^*)^n \longrightarrow \mathbb{Q}^n$

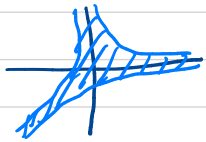
$\mathrm{trop}(Y) := \overline{\mathrm{val}(Y(K))} \subset \mathbb{R}^n$



- $\mathrm{Trop}(Y)$ = asymp. directions along which Y approaches infinity (exponentially).

  also called Logarithmic limit set.

<u>Ex.</u> $T = (\mathbb{C}^*)^2$ $\quad Y = \{x+y+1=0\} \rightsquigarrow$ line

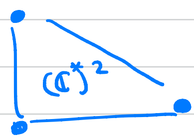
$\mathrm{Log}_t(Y) =$  $\quad \mathrm{trop}(Y) =$ 

Relation with toric varieties : $|\Sigma| = \mathrm{trop}(Y)$

X_Σ is the "smallest" toric variety s.t. $\overline{Y} \subset X_\Sigma$ is compact/complete.

$\mathbb{P}^2 \setminus \{3 \text{ pts}\}$

$(0:0:1)$ $\quad (0:1:0)$ $\quad (1:0:0)$



- Some properties of amoebas :
  - $\mathrm{Log}_t(Y)$ closed.
  - Every Conn. Comp. of $\mathbb{R}^n \setminus \mathrm{Log}_t(Y)$ is Convex.
  - Ronkin function $\rightsquigarrow$ Convex function & affine on each conn. comp. of the amoeba. complement.

<u>Extension to non-abelian case</u>

- $G = GL_n(\mathbb{C})$ $\quad K = U(n)$ max. comp. subgp.

- More generally one can consider a <u>reductive alg. gp. $G/\mathbb{C}$</u>.

  e.g. $G = GL_n(\mathbb{C}), SL_n(\mathbb{C}), SO_n(\mathbb{C}), Sp_{2n}(\mathbb{C}), \dots$

- <u>Eliyashev</u>: Geo. of generalized amoebas (2016).

  I don't know the relation with the previous construction.

- $G \times G \curvearrowright G$ Cartan decomp. describes $K \times K$-orbit space of G.

<u>Beyond the group case</u>

$G \curvearrowright X$ variety $T \curvearrowright X$

Name nothing to do with sphere! Comes from sph. functions / sph. harmonics

<u>Def.</u> X is a <u>spherical</u> G-variety if a <u>Borel</u> subgp. has a dense orbit.

For symp. geo.: $G \curvearrowright X \hookrightarrow \mathbb{P}(V)$ (compact, smooth)

K-inv. Hermitian product $\langle \cdot, \cdot \rangle$ on V

ω Kähler form induced from $\langle \cdot, \cdot \rangle$

Facts (M. Brion) $\rightsquigarrow$ "Sur l'image appl. moment...."

- X sph. iff X multi-free K-space (Symp. reds. = pt.)
- $K \backslash X \cong$ Kirwan polytope $= \mu(X) \cap \mathfrak{t}^*_+$

Book D. Timashev

$G \curvearrowright G/_H$ = homog. space

Ex.

- $SL_2(\mathbb{C})/T$

- $G \times G \curvearrowright G$ $\quad G = (G\times G)/G_{diag.} = \{(g,g)\}$

- $SL_n(\mathbb{C})/SO_n(\mathbb{C})$ = space of quadrics

- $G/_U$ e.g. $GL_n(\mathbb{C})/\left\{\begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1\end{bmatrix}\right\}$

  max. unipotent subgp.

- $\underset{\overset{\shortparallel}{G}}{T} \curvearrowright T$ $\quad H = \{1\}$

$$\left[\begin{array}{l} \cdot \; SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) / SL_2(\mathbb{C})_{diag.} \\ \quad {}_K\backslash G/_H = \text{space of hyperbolic triangles} \end{array}\right]$$

✓ Sph. (non-Arch) tropicalization ⇝ Luna-Vust, Tevelev-Vogiannou, K.-Manon

$Y \subset G/_H$

$\mathrm{val}: G/_H(K) \longrightarrow \mathcal{V}_{G/_H}$ = valuation cone

? sph. (Arch.) log. map.

• $G \curvearrowright G/H$ sph. homog. space

(G-equiv. Compactification $\rightsquigarrow$ Luna-Vust theory)

valuation co

val: $(K^*)^n \longrightarrow \mathbb{Q}^n$ $\rightsquigarrow$ val: $G/H\,(K) \longrightarrow \mathcal{V}_{G/H} \subset \mathcal{Q}_{G/H}$

$= T(K)$

$\mathcal{V}_{G/H} \downarrow$ co-simplicial cone; $\mathcal{Q}_{G/H} \downarrow$ r-dim $\mathbb{Q}$-v.s.

$$\mathcal{V}_{G/H} = \{ G\text{-inv. val. } v : \mathbb{C}(G/H) \to \mathbb{Q} \}$$

<u>Batyrev</u>: describe $K \backslash G/H$

$\to$ stated in [BHHK]

<u>Batyrev's Conj.</u>: $K \backslash G/H$ is a stratified manifold with corners where the boundary strata are in natural bijection with the faces σ of the val. cone $\mathcal{V}_{G/H}$ & one can recover K-stabilizer of each strata as the max. comp. subgp. in the <u>satellite subgp.</u> of σ.

Toric case:

$$\begin{tikzcd}
 & \mathbb{P}(V) & \\
X \arrow[ur, hook] \arrow[rr, "\mu"] & & \Delta(X) \\
(\mathbb{C}^*)^n = T \arrow[u, hook] \arrow[rr, "\mathrm{Log}"'] & & \mathbb{R}^n \arrow[u, hook]
\end{tikzcd}$$

Kirwan polytope

G/H

(related by a Legendre transform)

Question: Can we define a sph. Log. map which extends the above diagram to G/H in place of T?

$$G/H \xrightarrow{?} \mathbb{R}^r = \mathcal{Q}_{G/H}$$

parametrizing K-orbit?

Akhiezer's sph. functions

$\lambda \in \Lambda_{G/H}$ = wt. of B-eigen functions in $\mathbb{C}(G/H)$. $\subset \Lambda_G$

lattice weight of G/H — wt lattice of G

- Suppose G/H quasi-affine $\longrightarrow$ $\mathbb{C}[G/H]$ multi-free G-module

$$\mathbb{C}[G/H] = \bigoplus_{\lambda \in \Lambda^+_{G/H}} V_\lambda$$

$V_\lambda \subset \mathbb{C}[G/H]$ $\quad \{f_{\lambda,i}\}$ orth. normal basis for V_λ w.r.t. K-inv. Herm. prod.

<u>Def.</u>

$$\phi_\lambda(x) = \sum_i |f_{\lambda,i}(x)|^2$$

Normalize ϕ_λ by requiring $\phi_\lambda(eH) = 1$.

(Akhiezer)

<u>Prop.</u> $\left(\mathbb{C}[G/H] \cdot \overline{\mathbb{C}[G/H]}\right)^K = \bigoplus_{\lambda \in \Lambda^+_{G/H}} \mathbb{C}\phi_\lambda$

[BHHK]

<u>Prop.</u> ϕ_λ, $\lambda \in \Lambda^+_{G/H}$ separate K-orbits in G/H.

+ other properties.

• $\Gamma = \{\lambda_1, \dots, \lambda_s\}$ gen. $\Lambda^+_{G/H}$ as a monoid / semigp. $\Rightarrow$

We propose :

$$\mathrm{sLog}_{\Gamma, t} : G/H \longrightarrow \mathbb{R}^s$$
$$x \longmapsto \left(\log_t \Phi_{\lambda_1}(x), \dots, \log_t \Phi_{\lambda_s}(x)\right)$$

as analogue of the $\log_t$ map on torus.

• We prove some partial results in regard to $\mathrm{sLog}_{\Gamma, t}$.

e.g. • $\lim_{t \to 0} \mathrm{sLog}_{\Gamma, t}(G/H) \supset \mathcal{V}_{G/H}$.

we expect equality to hold

4.1. Two small examples with $G = \mathrm{SL}_2(\mathbb{C})$.

4.1.1. *The case of* $X = \mathrm{SL}_2(\mathbb{C})/T = (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta$. Let $G = \mathrm{SL}_2(\mathbb{C})$ and let $H = T$, the maximal (diagonal) torus of G. Consider the action of G on $\mathbb{P}^1$ by left multiplication, and the corresponding diagonal action of G on $\mathbb{P}^1 \times \mathbb{P}^1$. Then it is not hard to see that the stabilizer of the point $x_0 = ([1:0],[0:1]) \in \mathbb{P}^1 \times \mathbb{P}^1$ is precisely T, and that the orbit of x_0 under the G-action is $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta$, where Δ denotes the diagonal copy of $\mathbb{P}^1$ in the direct product. The compact group K is $SU(2)$ in this case. The (diagonal) action of $SU(2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ has moment map

$$\Phi([z],[w]) = \frac{i}{2}\left(\frac{zz^*}{\|z\|^2} + \frac{ww^*}{\|w\|^2}\right) - \left(\frac{i}{4\|z\|^2}\mathrm{tr}(zz^*) + \frac{i}{4\|w\|^2}\mathrm{tr}(ww^*)\right) \cdot I_{2\times 2} \in \mathfrak{su}(2)^*$$

where $([z],[w])$ are homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$ (so z, w are considered as elements of $\mathbb{C}^2$) and $I_{2\times 2}$ denotes the 2×2 identity matrix. Composing this with the map which quotients by the coadjoint action of $SU(2)$ on $\mathfrak{su}(2)^*$, i.e. the so-called "sweeping map" $\mathfrak{su}(2)^* \to \mathbb{R}_{\geq 0} = \mathfrak{su}(2)/SU(2)$, yields that the Kirwan map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{R}_{\geq 0}$ is given by

$$([z],[w]) \longmapsto \frac{1}{2}\sqrt{\left(\frac{|z_1|^2}{\|z\|^2} + \frac{|w_1|^2}{\|w\|^2} - 1\right)^2 + \left(\frac{|z_1|^2|z_2|^2}{\|z\|^4} + \frac{2\mathrm{Re}(z_1\overline{w}_1\overline{z}_2 w_2)}{\|z\|^2\|w\|^2} + \frac{|w_1|^2|w_2|^2}{\|w\|^4}\right)}.$$

The Kirwan polytope (i.e. the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under the above map) is the interval $[0, \frac{1}{2}]$ and the image of the diagonal is straightforwardly computed to be $\frac{1}{2}$, so the distinguished K-orbit (the diagonal) corresponds to the boundary value.

There is also another natural parametrization of K-orbits which can be described in terms of the angle between two complex lines in $\mathbb{C}^2$. More specifically, given $([z],[w]) \in \mathbb{P}^1 \times \mathbb{P}^1$ where $z, w \in \mathbb{C}^2 \setminus \{0\}$, we may define

$$\rho([z],[w]) := 1 - \frac{|\langle z, w\rangle|^2}{\|z\|^2\|w\|^2},$$

where $\langle z, w\rangle$ is the standard Hermitian product on $\mathbb{C}^2$. Geometrically, ρ is the quantity $\sin^2(\theta)$ where θ is the angle between the two complex lines spanned by z and w, or equivalently, the spherical distance between two distinct points $p, q \in \mathbb{P}^1 = S^2$ if we represent p, q by complex vectors $z, w \in \mathbb{C}^2$ with $\|z\| = \|w\| = 1$. It turns out that $\rho([z],[w]) = \rho([z'],[w'])$ if and only if the two pairs $([z],[w]), ([z'],[w'])$ are in the same $K = \mathrm{SU}(2)$-orbit, so the function ρ provides a parametrization of K-orbits in G/T. It follows from the Cauchy-Schwarz inequality that the image of ρ is the interval $(0,1]$. Thus, composing ρ with $-\log$, i.e. $([z],[w]) \longmapsto -\log(\rho([z],[w]))$, we obtain an identification of the K-orbit space with the valuation cone (which in this case is $\mathbb{R}_{\geq 0}$).

Finally, there is also our spherical function ϕ_2 corresponding to the irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of highest weight 2 in $\mathbb{C}[\mathrm{SL}_2(\mathbb{C})/T]$. This can be computed to be

$$\phi_2([z],[w]) = \frac{1}{|z_1w_2 - z_2w_1|^2}\left(|z_1w_1|^2 + \frac{1}{2}|z_1w_2 + w_1z_2|^2 + |z_2w_2|^2\right).$$

One can show that $\phi_2([z],[w]) \geq 1/2$. On the other hand, $\phi_2((1:1),(-1:1)) = 1/2$. This shows that the image of ϕ_2 is $[1/2, \infty)$. Hence we have a parametrization of the K-orbits by the points in this interval. One can obtain a parametrization of the K-orbit space by $\mathbb{R}_{\geq 0}$ (the valuation cone) by taking the limit, as $t \to 0$, of the image of $-\log_t(\phi_2)$.

The above computations show that, in this case, we have three parametrizations of the space of K-orbits $SU(2)\backslash SL_2(\mathbb{C})/T$ by (half open) intervals in $\mathbb{R}$, namely: the spherical function ϕ_2 above, the Kirwan map computed above, and the map ρ as above.

4.1.2. *The case of* $X = \mathrm{SL}_2(\mathbb{C})/N(T)$. As in the last section, we take $G = \mathrm{SL}_2(\mathbb{C})$ but this time we take $H = N(T)$. In this situation we have $N(T)/T \cong \mathbb{Z}/2\mathbb{Z}$ so there is a natural map $\mathrm{SL}_2(\mathbb{C})/T \to \mathrm{SL}_2(\mathbb{C})/N(T)$. The homogeneous space $\mathrm{SL}_2(\mathbb{C})/N(T)$ can be identified with $\mathbb{P}^2 \setminus Q$ where Q is a smooth conic, as follows: consider the map $\mathrm{Sym}^1(\mathbb{C}^2) \times \mathrm{Sym}^1(\mathbb{C}^2) \to \mathrm{Sym}^2(\mathbb{C}^2)$ given by multiplication. Note that $\mathrm{Sym}^1(\mathbb{C}^2) \cong \mathbb{C}^2$ and $\mathrm{Sym}^2(\mathbb{C}^2) \cong \mathbb{C}^3$. This product map induces a morphism $(\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta \to \mathbb{P}^2 \setminus Q$, where Q is the smooth conic defined by the vanishing of the discriminant on $\mathrm{Sym}^2(\mathbb{C}^2) \cong \mathbb{C}^3$. One sees that the natural projection $\mathrm{SL}_2(\mathbb{C})/T \to \mathrm{SL}_2(\mathbb{C})/N(T)$ is then identified with $(\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta \to \mathbb{P}^2 \setminus Q$. The non-identity element in the quotient $N(T)/T \cong \mathbb{Z}/2\mathbb{Z}$ corresponds to the involution on $\mathbb{P}^1 \times \mathbb{P}^1$ exchanging the two factors. This