RCFT correlators as equivalences of modular functors

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2d Rational Conformal Field Theory

VOA $\mathcal{V}$ $\xmapsto{\mathrm{Rep}(-)}$ modular fusion category $\mathcal{C}$

w/ finiteness conditions

- spherical fusion + non-degenerate braiding
- invertible ($\Rightarrow$ fully dualizable) object in a 4-category $\mathrm{BrTen} = \mathrm{Alg}_2(\mathrm{Pr}^L)$ "even higher Morita"
- Cobordism hypothesis : invertible, fully extended 4d TFT Crane-Yetter
- boundary theory : Reshetikhin-Turaev "framing anomaly"

(Chern-Simons)

$$\mathrm{RT}_{\mathcal{C}} : \mathrm{Bord}_{3-2-1}^{p_1,\, \mathrm{rib}(\mathcal{C})} \longrightarrow 2\mathrm{Vect}$$

linear categories

Full Conformal Blocks

described by an open-closed modular functor (symm. mon. 2-functors)

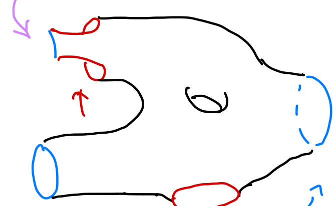
linear categories, profunctors, nats

$$\mathrm{Bl}_{\mathcal{C}} : \mathrm{Bord}_2^{o/c} \longrightarrow \mathrm{Prof}_{\Bbbk}$$

$$I \longmapsto \mathcal{C}$$ boundary field objects

$$S^1 \longmapsto Z(\mathcal{C})$$ bulk field objects

Drinfeld center



cut-and-paste boundary

free boundaries

Note:

$$\int_{S^1} \mathcal{C} \simeq Z(\mathcal{C})$$

for pivotal fusion categories

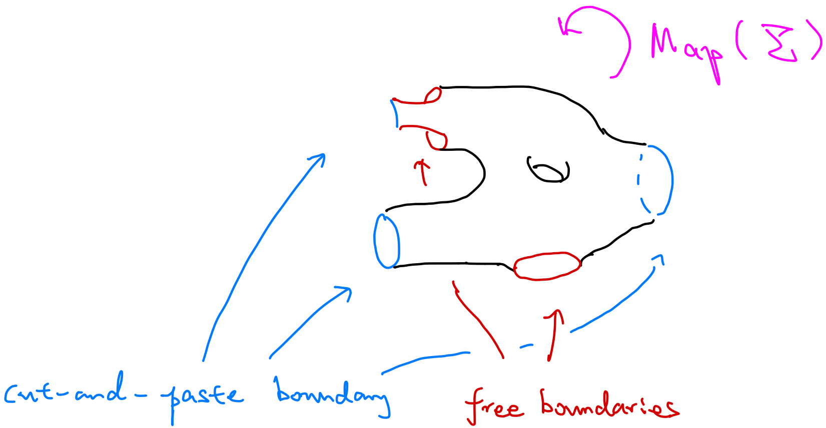
Full Conformal Blocks

$$\mathrm{Bl}_{\mathcal{C}} : \mathrm{Bord}_2^{o/c} \longrightarrow \mathrm{Prof}_k$$

$$\mathrm{IWS}^1 \xrightarrow[\shortmid]{\Sigma} S^1 \longmapsto \mathrm{Bl}_{\mathcal{C}}(\Sigma) : \mathcal{C} \otimes Z(\mathcal{C}) \xrightarrow{\shortmid} Z(\mathcal{C})$$

$$\| $$

$$\mathcal{C}^{op} \otimes Z(\mathcal{C})^{op} \otimes Z(\mathcal{C}) \longrightarrow \mathrm{Vect}_k$$



$$\mathrm{Bl}_{\mathcal{C}}(\Sigma, X \otimes Y \otimes Z) \in \mathrm{Vect}_k^{(f.d.)}$$

fixed field objects

with $\mathrm{Map}(\Sigma)$ action

functoriality / excision:

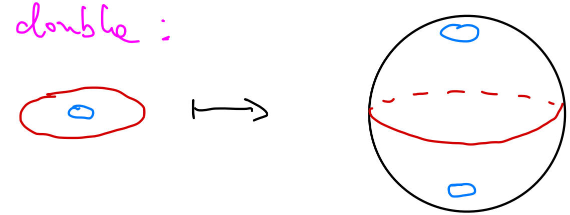
$$\mathrm{Bl}_{\mathcal{C}}(\Sigma \circ_{\ell} \Sigma') = \mathrm{Bl}_{\mathcal{C}}(\Sigma) \otimes_{\mathrm{Bl}_{\mathcal{C}}(\ell)} \mathrm{Bl}_{\mathcal{C}}(\Sigma')$$

← coend

- Bl_e extends closed MF restricted from 3d TFT

$$\mathrm{Bl}_e\big|_{\text{closed}} \simeq \mathrm{RT}_{Z(e)} \simeq \mathrm{RT}_e \circ \widehat{(\;-\;)}$$

double :



- they are anomaly free

- other realizations : $\mathrm{TV}_e \simeq \mathrm{SN}_e \simeq \mathrm{RT}_{Z(e)}$

state-sum | string-net

Turaev - Viro - Barrett - Westbury | Levin - Wen - Kirillov Jr.

for fixed field objects, space of (full) conformal blocks

$$\circlearrowleft \; \mathrm{Map}(\Sigma)$$

$$Bl_e(\Sigma, X \otimes Y \otimes Z)$$

$$\updownarrow \quad \text{Riemann - Hilbert correspondence}$$

$$\begin{array}{c} H_{X,Y,Z} \\ \downarrow p \\ \mathcal{M}_{g,n} \end{array}$$

Local System of horizontal sections of the vector bundle

with projectively flat KZ connection $\nabla = d + L_{-1} \otimes dz$

Correlators:

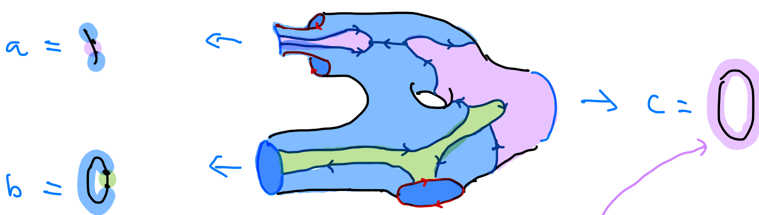
- global sections
- $\mathrm{Map}(\Sigma) = \pi_1(\mathcal{M}_{g,n})$ - invariant vector in $Bl_e(\Sigma, X \otimes Y \otimes Z)$

Consistent System of Correlators

$\{$ world sheets $\}$ $\xrightarrow{\mathrm{Cor}(-)}$ $\{$ conformal blocks $\}$

with topological defects

consistency $\begin{cases} \cdot \text{ invariance} \\ \cdot \text{ compatible w/ gluing} \end{cases}$



$\mapsto$ $\mathrm{Cor}(S)$ invariant under a subgroup of $\mathrm{Map}(\Sigma_S)$

$\cap$

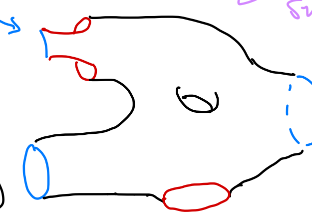
D-stratified 1-mfld

label system : defect bicategory $\mathcal{D}$

- obj : phases
- 1-mor : line defects
- 2-mor : point defects

underlying surface

$Bl_{\mathcal{C}}\Big($ $F(a) \in \mathcal{C}$, $F(b) \in Z(\mathcal{C})$



$\leftarrow F(c) \in Z(\mathcal{C})$ $\Big)$

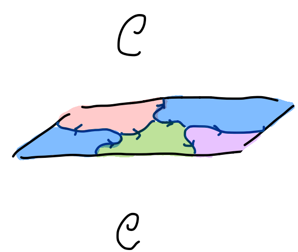
TFT-Construction of RCFT Correlators

[Fjelstad, Fuchs, Runkel, Schweigert, early 2000s]

- used $RT_{\mathcal{C}}(\hat{-})$ for the modular functor

- insight from Kapustin & Saulina:

$$\{\text{correlators}\} \hookrightarrow \{\text{surface defects in } RT_{\mathcal{C}}\}$$

$RT_{\mathcal{C}}\Big($ 

$\Big)$

this determines the defect bicategory [Fuchs, Schweigert, Valentino]

Eilenberg-Watts

$$\mathcal{D} = \mathcal{C}_{orb} := \mathrm{Fr}(\mathcal{C}) \simeq \mathcal{C}\text{-}\mathrm{Mod}^{tr}$$

orbifold completion

"cyclic associative algebras"

- obj. Δ-separable, symm. Frobenius alg. · $\mathcal{C}$ − module categories w/ traces

$$A \longmapsto \mathcal{M} = \mathrm{mod}\text{-}A$$

- hom. $\mathrm{Fr}(\mathcal{C})(A, B) := A\text{-}\mathrm{mod}\text{-}B$ · $\mathrm{Fun}^{rex}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$

$$X \longmapsto (-) \otimes_A X$$

· Q-systems

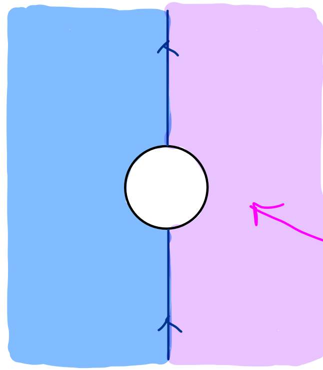
· encodes 2d Pachner moves  , ...

Field Contents

$X, Y \in A\text{-mod-}B$

$G = (-) \otimes_A Y$

$\mathcal{M} = \text{mod-}A$



$\mathcal{N} = \text{mod-}B$

$F = (-) \otimes_A X$

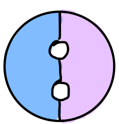
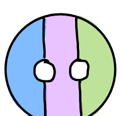
defect field object:

$$\mathbb{D}^{X,Y} = \underline{\mathrm{Nat}}(F, G) \in \mathcal{Z}(\mathcal{C})$$

monoidal bicategory

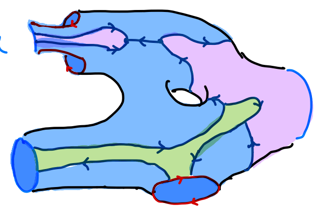
$\Downarrow$

operator products:

- Cor() $\Rightarrow$ vertical composition
- Cor() $\Rightarrow$ horizontal composition

$\Rightarrow$ $\mathcal{Z}(\mathcal{C})$-Cat enrichment of the defect bicategory $\mathcal{C}_{orb}$, with braided interchange law

Summary

$S =$ 

underlying bordism Σ_g

field contents

$\mathbb{F}a \in Bl_{\mathcal{C}}(I) \simeq \mathcal{C}$

$\mathbb{F}b, \mathbb{F}c \in Bl_{\mathcal{C}}(S^1) \simeq Z(\mathcal{C})$

Space of conformal blocks

$Bl_{\mathcal{C}}(\Sigma_g ; \mathbb{F}a, \mathbb{F}b, \mathbb{F}c)$

$\cup$

$Cor(S)$

labeled by defect bicategory

$\mathcal{C}_{orb} = Fr(\mathcal{C})$

$\simeq \mathcal{C}\text{-}Mod^{tr}$

- TFT construction : $\widehat{RT_{\mathcal{C}}(-)} \simeq RT_{Z(\mathcal{C})}$ [FFRS]
- String-net construction : $SN_{\mathcal{C}}$

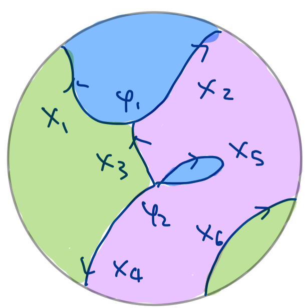
[Fuchs, Schweigert, Y]

- geometric $Map(\Sigma)$-action
- easier computation
- clarifies structures

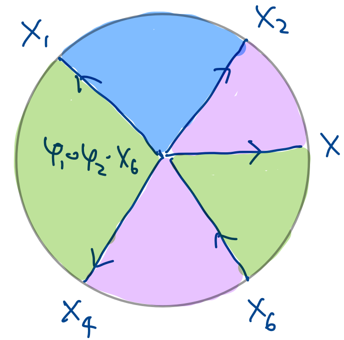
Graphical Calculus for Pivotal Bicategories

- pivotal bicategories : all 1-morphisms have both left and right adjoint and are coherently identified : $X^\vee \simeq {}^\vee X$

- calculus of oriented graphs on oriented disks :



$\longmapsto$



String-net construction for pivotal bicategories

[Fuchs, Schweigert, Y]

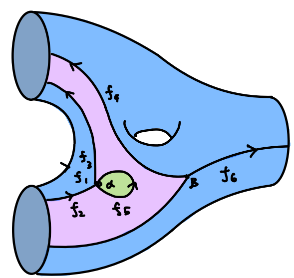
Let $\mathcal{B} \in \mathrm{PivBicat}$

category of $\mathcal{B}$-graphs on Σ, mor's are disk replacements

$$\mathrm{sn}_{\mathcal{B}}(\Sigma, b) := \mathrm{colim}\left(\mathrm{Graphs}_{\mathcal{B}}(\Sigma, b) \longrightarrow \mathrm{Vect} \right)$$

gluing boundary value

$$= k\left\{ \right. \quad \left. \right\} \Big/ \langle \text{disk replacements} \rangle$$



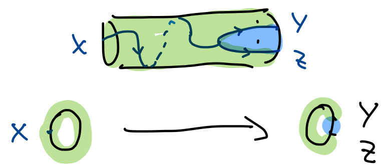
Map(Σ)

$$\mathrm{SphFus} \hookrightarrow \mathrm{PivTens} \xhookrightarrow{\text{deloop}} \mathrm{PivBicat}$$

recover Levin-Wen-Kirillov Jr. string-nets

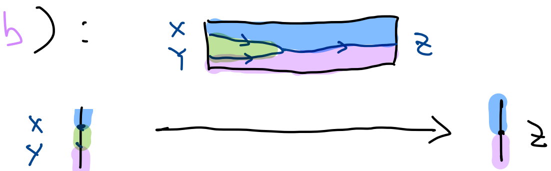
Cylinder categories :

· $sn_B(S^1)$:



for $a, b \in B$,

$sn_B(I; a, b)$:



$\simeq B(a, b)$

Cauchy completion $\longrightarrow$

· $SN_{\bar{B}}(S^1)$

· $SN_{\bar{B}}(I; a, b)$

$\simeq \bar{B}(a, b)$

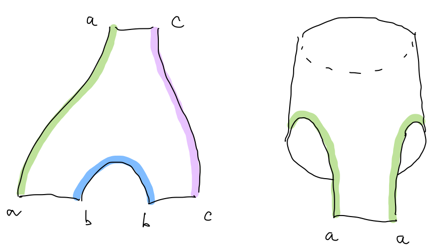
String-net modular functors:

- for each $a \in \mathcal{B}$, we have an open-closed MF

$$SN_{\mathcal{B},a} : \begin{cases} S^1 \mapsto SN_{\mathcal{B}}(S^1) \\ I \mapsto SN_{\mathcal{B}}(I; a, a) \simeq \bar{\mathcal{B}}(a,a) \end{cases}$$

a selection of trivial phase

- $\mathcal{B}$ itself gives an open-closed MF with boundary conditions obj $\mathcal{B}$



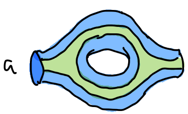
$\Rightarrow$

- (pseudo) Calabi-Yau object in Prof
- with "topological cyclic homology" $= SN_{\mathcal{B}}(S^1)$

- $SN_{e,\mathbb{1}} \equiv SN_e \simeq RT_{Z(e)} \simeq Bl_e$

gives MF of Conformal blocks

- $SN_{e_{orb},\mathbb{1}} \equiv SN_{e_{orb}}$ gives MF of World sheets WS_e

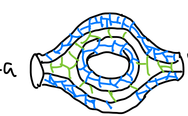
- correlators : $SN_{e_{orb}}(\Sigma; a, b) \ni$ 

Linear map (natural) $\longrightarrow$ Cor

$\Uparrow$

disk replacement does not change correlator !

Categorical symmetry

$SN_e(\Sigma, \mathbb{F}a, \mathbb{F}b) \ni$ 

functorial assignment of field contents

a functorial description of correlators and field contents

urges us to consider <u>(pseudo-) double categories</u>

$\mathbb{A} \in \mathrm{Dbl}$

$$\begin{tikzcd} a \arrow[r, "\shortmid" marking] \arrow[d] \arrow[dr, phantom, "\alpha"] & b \arrow[d] \\ c \arrow[r, "\shortmid" marking] & d \end{tikzcd}$$

- two kinds of 1-morphisms :

  horizontal $\cdot \xrightarrow{\shortmid} \cdot$ vertical $\begin{array}{c}\cdot\\ \downarrow\\ \cdot\end{array}$

- 2-cells compose in two directions

- notions of functors, transformations, and modifications and symmetric monoidal structures both $\mathrm{Bord}^{\mathrm{or}}_{2,\mathrm{o/c}}$ and Prof can be enhanced into

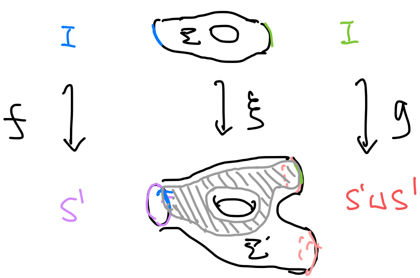
symm. mon. double categories: as "horizontal bicategory"

$\mathbb{B}\mathrm{ord}^{\mathrm{or}}_{2,\mathrm{o/c}}$

h-1mor: bordisms

v-1mor: embeddings

$$\begin{tikzcd} I \arrow[r, "\Sigma"] \arrow[d, "f"'] \arrow[dr, phantom, "\xi"] & I \arrow[d, "g"] \\ S^1 \arrow[r, "\Sigma'"'] & S^1 \sqcup S^1 \end{tikzcd}$$



$\mathbb{P}\mathrm{rof}$

profunctor

functor

$$\begin{tikzcd} A \arrow[r, "P"] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "G"] \\ C \arrow[r, "Q"'] & D \end{tikzcd}$$

$$P(-,\sim) \xRightarrow{\alpha} Q(F(-), G(\sim))$$

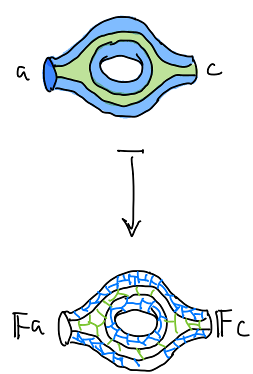
Fields and correlators $\Longrightarrow$ relative field theory in $\otimes$-Dbl

$$\begin{tikzcd} \mathbb{B}\mathrm{ord} \arrow[rr, bend left, "\mathbb{WS}_e = \mathfrak{S}\mathcal{N}_{e_{orb}}"] \arrow[rr, bend right, "\mathbb{Bl}_e = \mathfrak{S}\mathcal{N}_e"'] \arrow[rr, phantom, "\Downarrow \mathcal{U}_*"] & & \mathbb{P}\mathrm{rof} \end{tikzcd}$$

$\mathcal{SN}_{\mathcal{B},a} \longmapsto \mathfrak{S}\mathcal{N}_{\mathcal{B},a}$

Components :

$$\begin{tikzcd} \mathbb{WS}_e(\ell) \arrow[r, "\mathbb{WS}_e(\Sigma)"] \arrow[d, "\mathbb{F}_\ell"'] \arrow[dr, phantom, "\mathrm{Cor} \Downarrow"] & \mathbb{WS}_e(\ell') \arrow[d, "\mathbb{F}_{\ell'}"] \\ \mathbb{Bl}_e(\ell) \arrow[r, "\mathbb{Bl}_e(\Sigma)"'] & \mathbb{Bl}_e(\ell') \end{tikzcd}$$

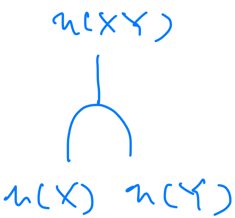


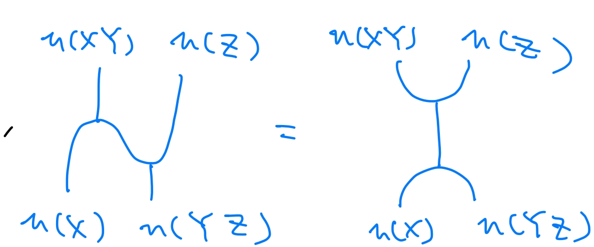
- well defined
  $\Rightarrow$ categorical symmetry
- vertically natural
  $\Rightarrow$ MCG – invariance
- horizontally functorial
  $\Rightarrow$ sewing constraints

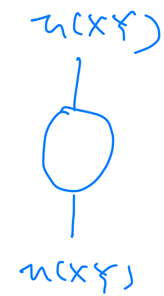
- $$\begin{tikzcd} \mathbb{B}\mathrm{ord} \arrow[rr, bend left, "\mathbb{W}\mathbb{S}_{\mathcal{C}} = \$N_{\mathcal{C}_{orb}}"] \arrow[rr, bend right, "\mathbb{B}\mathbb{l}_{\mathcal{C}} = \$N_{\mathcal{C}}"'] \arrow[rr, phantom, "\Downarrow u_*"] & & \mathbb{P}\mathrm{rof} \end{tikzcd}$$ is induced by $u : \mathcal{C}_{orb} \longrightarrow \mathcal{C}$

sending $\begin{tikzcd} X \arrow[r, bend left, "A"] \arrow[r, bend right, "B"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd}$ to $\begin{tikzcd} \dot{X} \arrow[r, bend left, "*"] \arrow[r, bend right, "*"'] \arrow[r, phantom, "\Downarrow\dot{\alpha}"] & \dot{Y} \end{tikzcd}$

which is a (rigid) separable Frobenius functor:

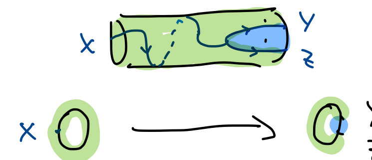
lax 

and colax, 

, 

- Theorem [FSY, in progress]: u_* is an equivalence!

$$\begin{tikzcd}
\mathbb{B}\mathrm{ord} \arrow[rr, bend left, "\mathbb{W}\mathrm{S}_{\mathcal{C}} = \mathcal{S}\mathrm{N}_{\mathcal{C}_{orb}}"] \arrow[rr, bend right, "\mathbb{B}\mathrm{l}_{\mathcal{C}} = \mathcal{S}\mathrm{N}_{\mathcal{C}}"'] \arrow[rr, phantom, "\mathcal{U}_* \Downarrow \wr"] & & \mathbb{P}\mathrm{rof}
\end{tikzcd}$$



- in particular, $\mathcal{S}\mathrm{N}_{\mathcal{C}_{orb}}(S^1) \simeq Z(\mathcal{C})$

- alternative description of conformal blocks (in RCFT):
  they are worldsheets up to categorical symmetries

Lax Biadjunction from Orbifold Completion

$\mathcal{B} \in \mathrm{PivBicat} \xmapsto{(-)} \mathcal{B}_{orb}$ ← oriented version of 2-idempotent completion

- obj : $\mathcal{A} \equiv (a, A)$, $a \in \mathcal{B}$, $A \in \mathrm{End}_{\mathcal{B}}(a)$

  Δ-Separable sym. Frob. monad

- $\mathcal{B}_{orb}(\mathcal{A}, \mathcal{B}) \subset \mathcal{B}(a, b)$, 1-, 2- morphs w/ bimodule str.

- Lax biadjunction :

$$\begin{tikzcd} \mathcal{B} \arrow[r, bend left, "\imath"] \arrow[r, phantom, "\perp_{lax}"] & \mathcal{B}_{orb} \arrow[l, bend left, "u"] \end{tikzcd}$$

$\begin{cases} \imath : a \longmapsto (a, 1_a) \\ u : (a, A) \longmapsto a \end{cases}$

both are Δ-Sep. Frob functors, $\imath$ is strong

$$\mathcal{B}_{orb}(\imath a, \mathcal{A}) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{B}(a, u\mathcal{A})$$

↖ instead of equivalence

Theorem (generalized) [Fuchs, Schweigert, Y]

The lax biadjunction
$$\begin{tikzcd} \mathcal{B} \arrow[r, bend left, "\imath"] \arrow[r, phantom, "\perp_{lax}"] & \mathcal{B}_{orb} \arrow[l, bend left, "u"] \end{tikzcd}$$
induces an adjoint equivalence in $\mathrm{DbL}^{\otimes}(\mathbb{B}\mathrm{ord}, \mathbb{P}\mathrm{rof})$ for any choice of $b \in \mathcal{B}$

$b \equiv (b, 1) \in \mathcal{B}_{orb}$

$$\begin{tikzcd}[column sep=huge] \mathbb{B}\mathrm{ord} \arrow[r, bend left=60, "SN_{\mathcal{B}_{orb}, b}"] \arrow[r, bend right=60, "SN_{\mathcal{B}, b}"'] \arrow[r, phantom, "u_* \Downarrow \ \cong \ \Uparrow \imath_*"] & \mathbb{P}\mathrm{rof} \end{tikzcd}$$

Rough idea

(trivial, as $1 = \mathcal{U} \circ \iota$)

- the lax biadjunction $\iota \dashv \mathcal{U}$ comes with unit and counit $\varepsilon : \iota\mathcal{U} \Rightarrow 1_{\mathcal{B}orb}$ that are lax and oplax
- the counit is compatible with the lax str. of the functors and partially compatible with the colax str.
- the 1-cell component of ε is quasi-invertible (in a precise sense)
- write down the properties in string-diagrams

$\Rightarrow$ invertible modification $\varepsilon_* : \iota_* \mathcal{U}_* \overset{\sim}{\Rrightarrow} 1_{SN_{\mathcal{B}orb,b}}$

Return to bicategorical equivalence

- $\mathbb{B}\mathrm{ord}$ and $\mathbb{P}\mathrm{rof}$ are so called

framed bicategories / proarrow equipments $\iff$ all vertical 1-mor. have both conjoints & companions

embedding
$$\begin{tikzcd} \mathcal{C} \arrow[d, "f"'] \\ \mathcal{C}' \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{C} \arrow[r, "\Sigma_f", "+"' marking, shift left] & \mathcal{C}' \arrow[l, "\Sigma^f", "+"' marking, shift left] \end{tikzcd}$$
bordisms,

functor
$$\begin{tikzcd} A \arrow[d, "F"'] \\ A' \end{tikzcd} \longmapsto \begin{tikzcd} A \arrow[r, "F_*", "+"' marking, shift left] & A' \arrow[l, "F^*", "+"' marking, shift left] \end{tikzcd}$$
profunctors

- By a result of Shulman & West-Hansen the diagram in $\otimes$-Dbl

$$\begin{tikzcd} \mathbb{B}\mathrm{ord} \arrow[r, bend left=50, "SN_{Borb,b}"{name=U}] \arrow[r, bend right=50, "SN_{B,b}"'{name=D}] & \mathbb{P}\mathrm{rof} \arrow[from=U, to=D, shorten=4pt, Rightarrow, bend right, "u_*"', shift right=3] \arrow[from=D, to=U, shorten=4pt, Rightarrow, bend right, "v_*"', shift right=3] \arrow[from=U, to=D, phantom, "\cong"] \end{tikzcd}$$

lifts to a diagram in $\otimes$-Bicat where the monoidal transformations are (necessarily) equivalences.

Outlook

- find a good generalization of this story to homotopical settings

- enriched, oriented $\mathcal{B}$-factorization homology ?