# Orbifold data
## as gaugeable non-invertible symmetries

**Nils Carqueville**

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based on joint work with Ilka Brunner, Catherine Meusburger, Vincentas Mulevičius, Daniel Plencner, Ana Ros Camacho, Ingo Runkel, Gregor Schaumann, Daniel Scherl, and **Lukas Müller**: arXiv:2307.06485 [math.QA]

overview: arXiv:2307.16674 [math-ph]

slides: `https://carqueville.net/nils/orbdat.pdf`

# In a nutshell

**Orbifold data** . . . are algebraic representations of Pachner moves

. . . are objects of a higher Morita category

. . . are special defects in defect TQFT

. . . are gaugeable (non-invertible) symmetries

. . . give rise to state sum models

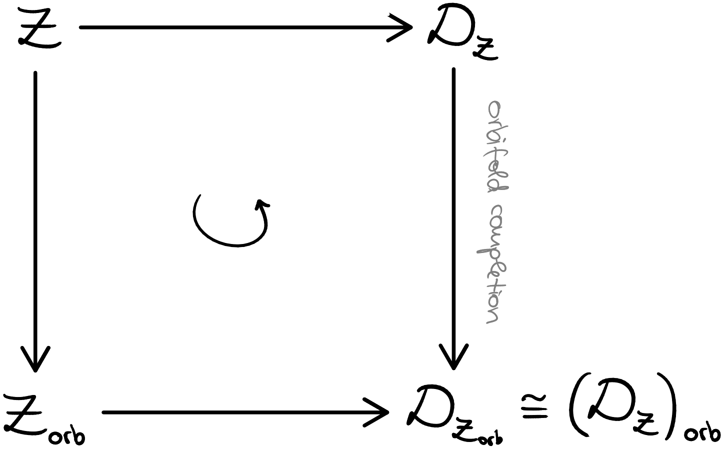
# In a nutshell

**Orbifold data** . . . are algebraic representations of Pachner moves
. . . are objects of a higher Morita category
. . . are special defects in defect TQFT
. . . are gaugeable (non-invertible) symmetries
. . . give rise to state sum models

**Theorem.** Let $\mathcal{T}$ be 3-category with duals. The higher Morita category $\mathcal{T}_{\mathrm{orb}}$ of orbifold data in $\mathcal{T}$ has duals.

**Theorem.** Let $\mathcal{Z}$ be 3d defect TQFT and $\mathcal{D}_{\mathcal{Z}}$ its 3-category with duals. From $(\mathcal{D}_{\mathcal{Z}})_{\mathrm{orb}}$ one obtains 3d **defect TQFT** $\mathcal{Z}_{\mathrm{orb}}$.

**Applications.**
- "Defect state sum models are orbifolds of the trivial defect TQFT."
- "Reshetikhin–Turaev defect TQFTs without thinking"
- "Douglas–Reutter 4-manifold invariants via orbifolds"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orbifold completion
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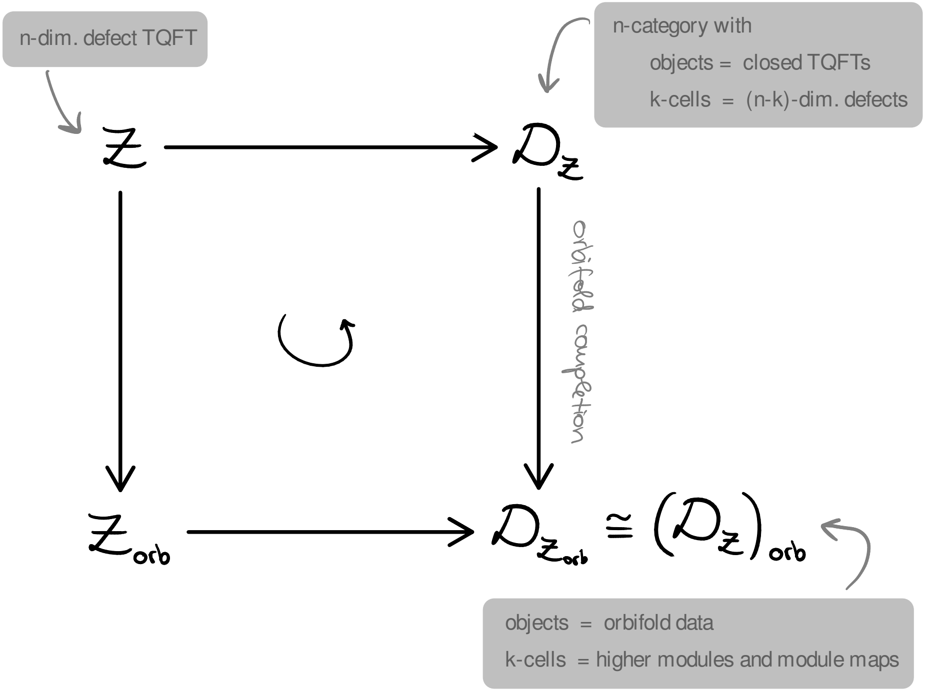

n-dim. defect TQFT
n-category with
objects = closed TQFTs
k-cells = (n-k)-dim. defects
$\mathcal{Z}$
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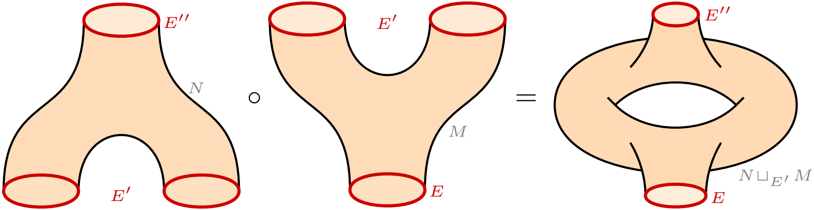
$\mathcal{Z}_{orb}$
$\mathcal{D}_{\mathcal{Z}_{orb}} \cong (\mathcal{D}_{\mathcal{Z}})_{orb}$
objects = orbifold data
k-cells = higher modules and module maps

# Closed TQFT

An **$n$-dimensional closed oriented TQFT** is symmetric monoidal functor

$$\mathrm{Bord}^{\mathrm{or}}_{n,n-1} \longrightarrow \mathcal{C}$$



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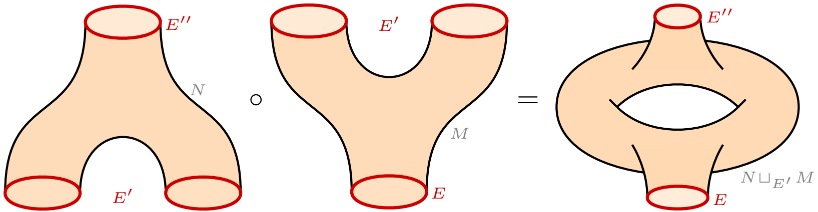



**Classification.**

(1d closed oriented TQFTs) $\cong$ (dualisable objects)

(2d closed oriented TQFTs) $\cong$ (commutative Frobenius algebras)

(3d closed oriented TQFTs) $\cong$ (J-algebras)

(4d closed oriented TQFTs) $\cong$ ??

Juhasz 2014

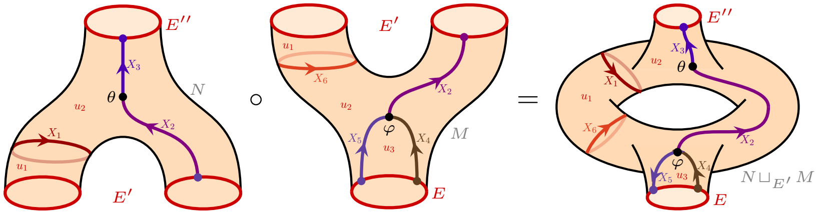
# Defect TQFT

An **$n$-dimensional defect TQFT** is symmetric monoidal functor

$$\mathrm{Bord}^{\mathrm{def}}_{n,n-1}(\mathbb{D}) \longrightarrow \mathcal{C}$$

depending on set of **defect data** $\mathbb{D}$ consisting of

- set $D_n$ of "bulk theories"
- sets $D_j$ of $j$-dimensional "defects" for $j \in \{0, 1, \dots, n-1\}$
- adjacency rules. . .




Davydov/Kong/Runkel 2011, Carqueville/Runkel/Schaumann 2017

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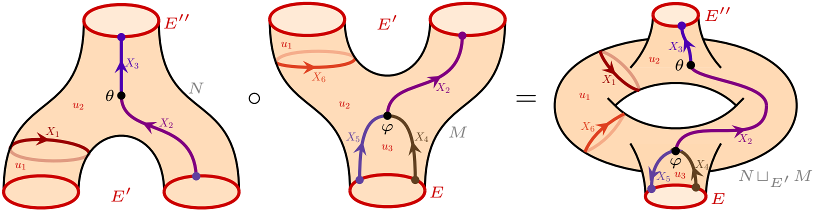



Non-full embedding $\mathrm{Bord}^{\mathrm{or}}_{n,n-1} \hookrightarrow \mathrm{Bord}^{\mathrm{def}}_{n,n-1}(\mathbb{D})$ for all $u \in D_n$

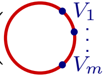
Davydov/Kong/Runkel 2011, Carqueville/Runkel/Schaumann 2017

# Examples of 2d defect TQFTs

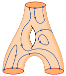
**Trivial defect TQFT** $\mathcal{Z}_2^{\mathrm{triv}} \colon \mathrm{Bord}_{2,1}^{\mathrm{def}}(\mathbb{D}^{\mathrm{triv}2}) \longrightarrow \mathrm{Vect}_{\Bbbk}$

$D_2^{\mathrm{triv}2} := \{\Bbbk\}$

$D_1^{\mathrm{triv}2} := \mathrm{Ob}(\mathrm{vect}_{\Bbbk})$

$\mathcal{Z}_2^{\mathrm{triv}}\Big($  $\Big) := V_1 \otimes \cdots \otimes V_m$


$D_0^{\mathrm{triv}2} := \mathrm{Mor}(\mathrm{vect}_{\Bbbk})$

$\mathcal{Z}_2^{\mathrm{triv}}\Big($  $\Big) :=$ (evaluate 0- und 1-strata as string diagrams in $\mathrm{vect}_{\Bbbk}$)

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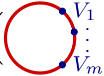
$\mathcal{Z}_2^{\text{triv}}\Bigg($ 

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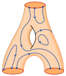
$\mathcal{Z}_2^{\text{triv}}\Bigg($  $\Bigg)$ := (evaluate 0- und 1-strata as string diagrams in $\operatorname{vect}_{\Bbbk}$)

**State sum models** $\mathcal{Z}_2^{\text{ss}}$:
separable symmetric Frobenius $\Bbbk$-algebras and bimodules

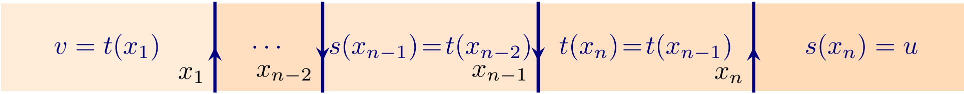
**B-twisted sigma models** $\mathcal{Z}^{\text{B}\sigma}$:
Calabi–Yau manifolds and their derived categories

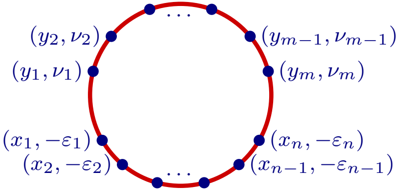
**Landau–Ginzburg models** $\mathcal{Z}^{\text{LG}}$:
isolated singularities and matrix factorisations

# Higher categories from defect TQFTs

**Theorem.** For $\mathcal{Z}\colon \mathrm{Bord}^{\mathrm{def}}_{2,1}(\mathbb{D}) \longrightarrow \mathcal{C}$, there is pivotal 2-category $\mathcal{D}_{\mathcal{Z}}$ with

- objects: elements of $D_2$
- 1-cells $X\colon u \longrightarrow v$ are lists of composable elements of $D_1$




- $\mathrm{Hom}(X,Y) = \mathcal{Z}\Bigg($  $\Bigg)$


- composition: “pair-of-pants with defects”

Davydov/Kong/Runkel 2011

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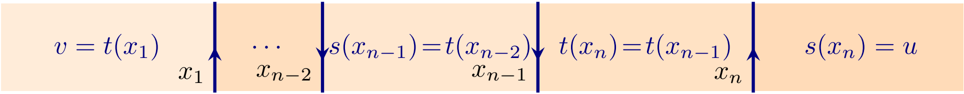
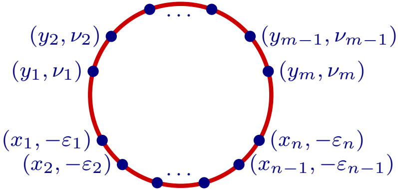



- $\mathrm{Hom}(X, Y) = \mathcal{Z}\Big($  $\Big)$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**Examples.**

$$\mathcal{D}_{\mathcal{Z}_2^{\mathrm{triv}}} \cong \mathrm{B}\,\mathrm{vect}_{\Bbbk}$$

$$\mathcal{D}_{\mathcal{Z}_2^{\mathrm{ss}}} \cong \mathrm{ssFrob}(\mathrm{vect}_{\Bbbk}) \cong \big(\mathcal{D}_{\mathcal{Z}_2^{\mathrm{triv}}}\big)^{\odot}_{\mathrm{orb}}$$

$$\mathcal{D}_{\mathcal{Z}_2^{\mathrm{LG}}} \cong \mathcal{LG}$$

Davydov/Kong/Runkel 2011

# Examples of 3d defect TQFTs

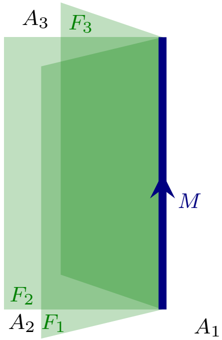
**Reshetikhin–Turaev defect TQFT** $\mathcal{Z}^{\mathrm{RT}}_{\mathcal{M}}$ for modular fusion category $\mathcal{M}$:

$D_3^{\mathrm{RT}} := \{$commutative $\Delta$-separable Frobenius algebras $A$ in $\mathcal{M}\}$

$D_2^{\mathrm{RT}} := \{\Delta$-sep. sym. Frobenius alg. $F$ with comp. bimodule structure$\}$

$D_1^{\mathrm{RT}} := \{$multimodules $M\}$

$D_0^{\mathrm{RT}} := \{$multimodule maps$\}$




**Trivial defect TQFT** $\mathcal{Z}_3^{\mathrm{triv}} \;\cong\; \mathcal{Z}^{\mathrm{RT}}_{\mathrm{vect}_{\Bbbk}}\Big|_{D_3^{\mathrm{RT}} \longrightarrow \{\Bbbk\}}$

Kapustin/Saulina 2010, Carqueville/Runkel/Schaumann 2017, Koppen/Mulevičius/Runkel/Schweigert 2021, Carqueville/Müller 2023

# Higher categories from defect TQFTs

**Theorem.** For $\mathcal{Z} \colon \operatorname{Bord}_{3,2}^{\mathrm{def}}(\mathbb{D}) \longrightarrow \mathcal{C}$, there is 3-category with duals $\mathcal{D}_{\mathcal{Z}}$:

- objects: elements of $D_3$
- $k$-cells: $(3-k)$-fold cylinders over defect $k$-balls, $k \in \{1,2\}$
- 3-cells: $\mathcal{Z}($"defect 2-sphere"$)$
- composition: "pair-of-pants with defects"
- duals: bending lines and surfaces

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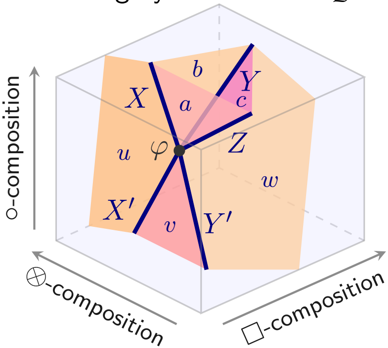

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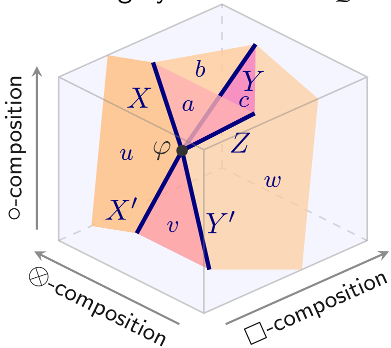


**Examples.**

$$\mathcal{D}_{\mathcal{Z}_3^{\mathrm{triv}}} \cong \mathrm{B}\,\mathrm{ssFrob}(\mathrm{vect}_{\Bbbk}) \cong \mathrm{B}\,\mathcal{D}_{\mathcal{Z}_2^{\mathrm{ss}}}$$

$$\mathcal{D}_{\mathcal{Z}_3^{\mathrm{ss}}} \cong \big(\mathcal{D}_{\mathcal{Z}_3^{\mathrm{triv}}}\big)^{\odot}_{\mathrm{orb}} \supset \mathrm{sFus}_{\Bbbk}$$

$$\mathcal{D}_{\mathcal{Z}^{\mathrm{RT}}_{\mathcal{M}}} \cong \big(\mathrm{B}\,\Delta\mathrm{ssFrob}(\mathcal{M})\big)\big)_{\mathrm{orb}}$$

Carqueville/Meusburger/Schaumann 2016, Barrett/Meusburger/Schaumann 2012, Carqueville/Müller 2023

# Examples of $n$-dimensional defect TQFTs

**Euler defect TQFT** $\mathcal{Z}^{\text{eu}}_{\Psi} \colon \operatorname{Bord}^{\text{def}}_{n,n-1} \longrightarrow \operatorname{Vect}_{\Bbbk}$, where

$\operatorname{Bord}^{\text{def}}_{n,n-1}$: stratified bordisms without labels
$\Psi = (\psi_1, \dots, \psi_n) \in (\Bbbk^\times)^n$

$$\mathcal{Z}^{\text{eu}}_{\Psi}(\text{object } E) := \Bbbk$$

$$\mathcal{Z}^{\text{eu}}_{\Psi}(\text{bordism } M) := \prod_{j=1}^{n} \prod_{j\text{-strata}\, \sigma_j \subset M} \psi_j^{\chi(\sigma_j) - \frac{1}{2}\chi(\partial\sigma_j)}$$

Quinn 1995, Carqueville/Runkel/Schaumann 2017

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**Euler completion** $\mathcal{Z}^{\odot}$ of any defect TQFT $\mathcal{Z}$ satisfies

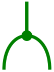
$$\left(\mathcal{Z}^{\odot}\right)^{\odot} \cong \mathcal{Z}^{\odot} \qquad \mathcal{Z}^{\odot} \otimes \mathcal{Z}^{\mathrm{eu}}_{\Psi} \cong \mathcal{Z}^{\odot}$$

**Euler completion** $\mathcal{D}^{\odot}_{\mathcal{Z}} \cong \mathcal{D}_{\mathcal{Z}^{\odot}}$ of higher defect categories

Quinn 1995, Carqueville/Runkel/Schaumann 2017

# $\Delta$-separable symmetric Frobenius algebras

$A \in \mathcal{C}$ with

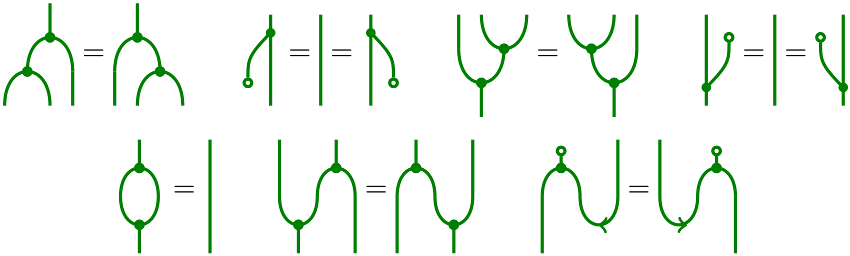
$\mu =$  $: A \otimes A \longrightarrow A$

 $: \mathbb{1} \longrightarrow A$

$\Delta =$  $: A \longrightarrow A \otimes A$

 $: A \longrightarrow \mathbb{1}$

such that



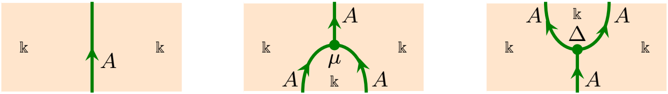
($A$ need *not* be commutative.)

# State sum models

<u>*Input*</u>: $\Delta$-separable symmetric Frobenius $\Bbbk$-algebra $(A, \mu, \Delta)$

(1) Choose oriented **triangulation** $t$ for every bordism $\Sigma$ in $\mathrm{Bord}_{2,1}^{\mathrm{or}}$

(2) **Decorate Poincaré dual** graph with $(\Bbbk, A, \mu, \Delta)$:




(3) Obtain $\Sigma^{t,A}$ in $\mathrm{Bord}_{2,1}^{\mathrm{def}}(\mathbb{D}^{\mathrm{triv}})$ and define $\mathcal{Z}_A(\Sigma) = \mathcal{Z}_2^{\mathrm{triv}}(\Sigma^{t,A})$

Fukuma/Hosono/Kawai 1992, Lauda/Pfeiffer 2006

## State sum models

*Input*: $\Delta$-separable symmetric Frobenius $\mathbb{C}$-algebra $(A, \mu, \Delta)$

(1) Choose oriented **triangulation** $t$ for every bordism $\Sigma$ in $\mathrm{Bord}_2$

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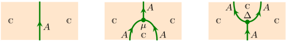


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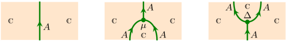


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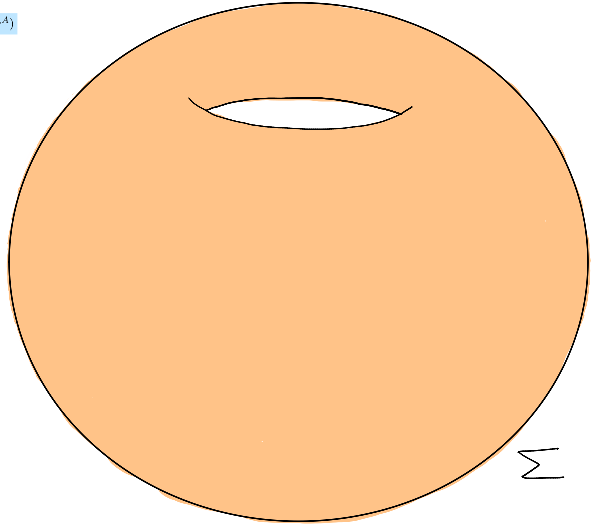

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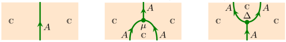


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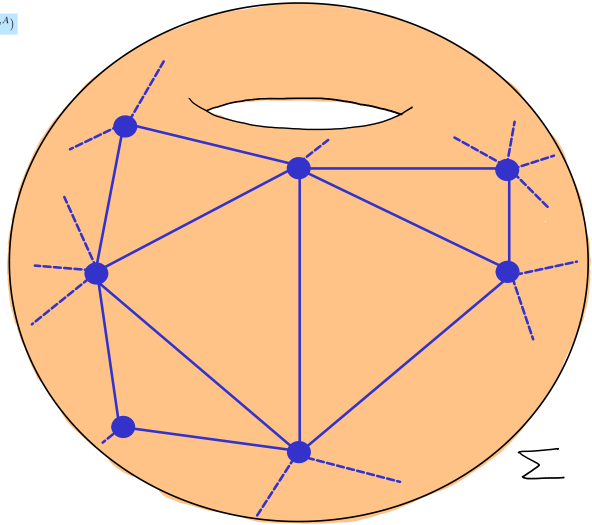

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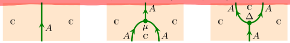


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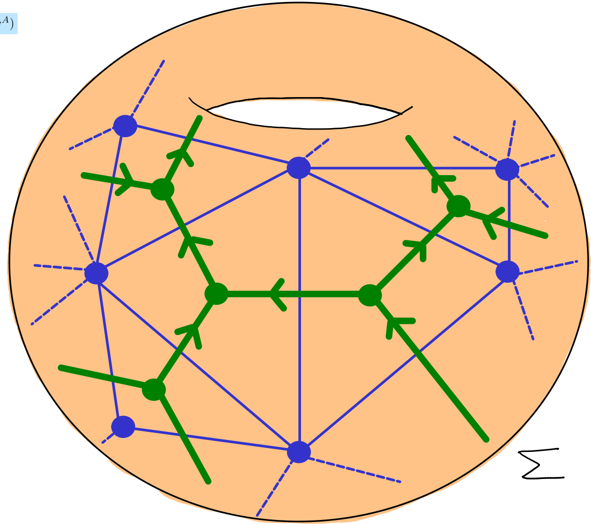

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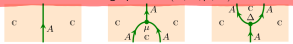


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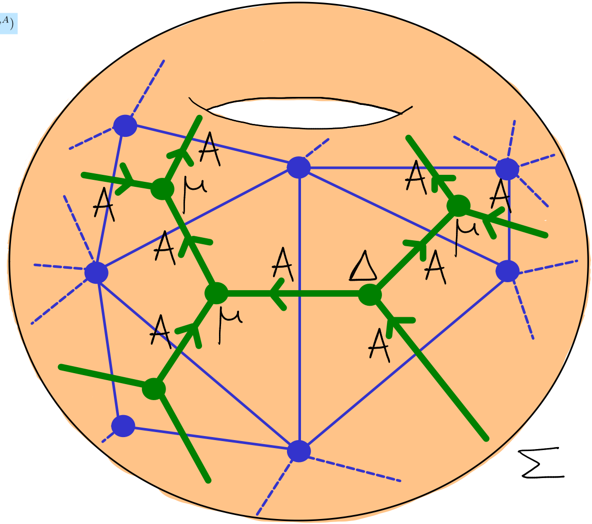

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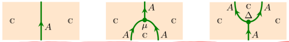


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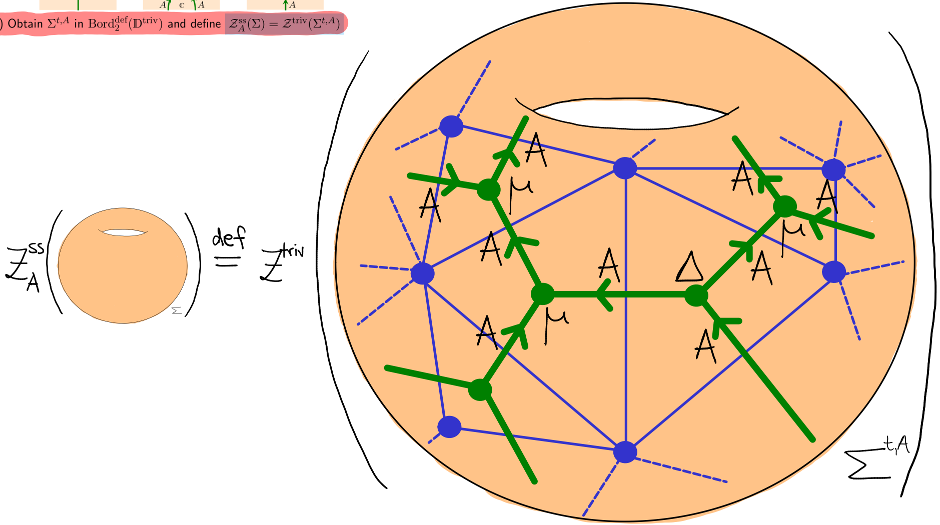

# State sum models

*Input*: $\Delta$-separable symmetric Frobenius $\Bbbk$-algebra $(A, \mu, \Delta)$

(1) Choose oriented **triangulation** $t$ for every bordism $\Sigma$ in $\mathrm{Bord}^{\mathrm{or}}_{2,1}$

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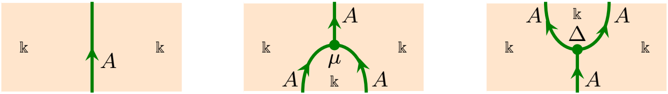


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**Theorem.** Construction yields TQFT $\mathcal{Z}_A \colon \mathrm{Bord}^{\mathrm{or}}_{2,1} \longrightarrow \mathrm{Vect}_{\Bbbk}$.

Fukuma/Hosono/Kawai 1992, Lauda/Pfeiffer 2006

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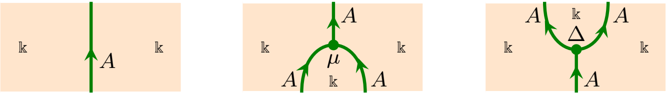


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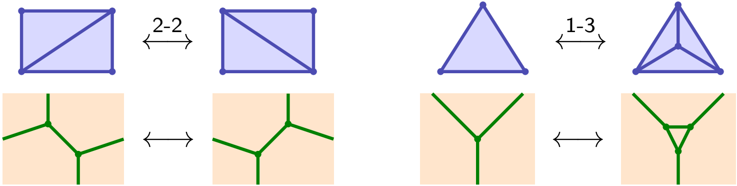
*Proof sketch*: Defining properties of $(A, \mu, \Delta)$ encode invariance under **Pachner moves** $\Longrightarrow$ independent of choice of triangulation:

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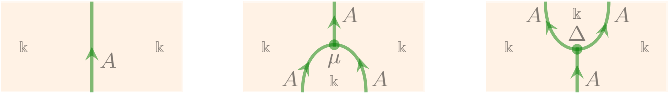


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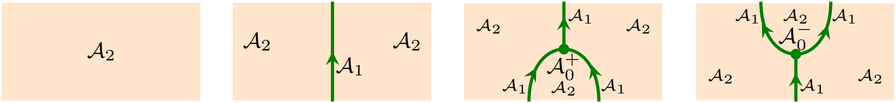
No need to consider only algebras over $\Bbbk$!

# Orbifolds

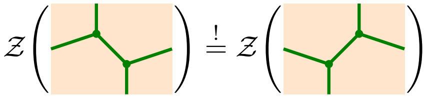
**Definition.** Let $\mathcal{Z}\colon \mathrm{Bord}^{\mathrm{def}}_{2,1}(\mathbb{D}) \longrightarrow \mathcal{C}$ be defect TQFT.

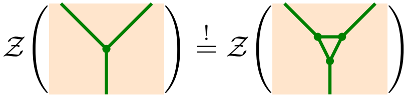
An **orbifold datum** for $\mathcal{Z}$ is $\mathcal{A} \equiv (\mathcal{A}_2, \mathcal{A}_1, \mathcal{A}_0^+, \mathcal{A}_0^-)$:




$\mathcal{A}_2 \in D_2$ $\quad$ $\mathcal{A}_1 \in D_1$ $\quad$ $\mathcal{A}_0^+ \in D_0$ $\quad$ $\mathcal{A}_0^- \in D_0$

such that (dual) *Pachner moves become identities* under $\mathcal{Z}$:





Carqueville/Runkel 2012, Fröhlich/Fuchs/Runkel/Schweigert 2009

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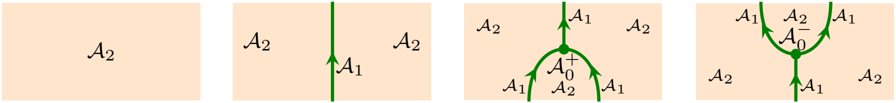


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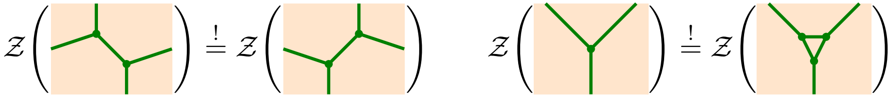

**Definition & Theorem.**
Triangulation + $\mathcal{A}$-decoration + evaluation with $\mathcal{Z}$ = **$\mathcal{A}$-orbifold TQFT**

$$\mathcal{Z}_{\mathcal{A}} \colon \mathrm{Bord}^{\mathrm{or}}_{2,1} \longrightarrow \mathcal{C}$$

Carqueville/Runkel 2012, Fröhlich/Fuchs/Runkel/Schweigert 2009

# Algebraic characterisation of orbifolds

**Theorem.**

2d defect TQFT $\mathcal{Z} \implies$ pivotal 2-category $\mathcal{D}_{\mathcal{Z}}$

**Lemma.**

$\{$orbifold data for $\mathcal{Z}\} \cong \{\Delta$-separable symmetric Frobenius algebras in $\mathcal{D}_{\mathcal{Z}}\}$

Davydov/Kong/Runkel 2011

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**Examples.**

- $\Delta$-separable symmetric Frobenius algebras in $\mathrm{BVect}_{\mathbb{k}}$
  $= \Delta$-separable symmetric Frobenius $\mathbb{k}$-algebras ☺
  $\implies$ $\mathcal{Z}_A = (\mathcal{Z}_2^{\mathrm{triv}})_A$ (*"State sum models are orbifolds of the trivial TQFT."*)
- A ***G*-action** in $\mathcal{D}_{\mathcal{Z}}$ is 2-functor $\rho\colon \mathrm{B}\underline{G} \longrightarrow \mathcal{D}_{\mathcal{Z}}$.

  **Lemma.** $A_G := \bigoplus_{g \in G} \rho(g)$ is $\Delta$-separable Frobenius algebra in $\mathcal{D}_{\mathcal{Z}}$.

  $\implies$ $G$-orbifolds are orbifolds: $\mathcal{Z}^G = \mathcal{Z}_{A_G}$ ☺

**Orbifolds unify** gauging of symmetry groups and state sum models.

Davydov/Kong/Runkel 2011, Fröhlich/Fuchs/Runkel/Schweigert 2009, Brunner/Carqueville/Plencner 2014

# There are many other orbifolds!

**Orbifold completion** of pivotal 2-category $\mathcal{B}$ is pivotal 2-category $\mathcal{B}_{\mathrm{orb}}$:

– objects: **$\Delta$-separable symmetric Frobenius algebras** $A \in \mathcal{B}(\alpha, \alpha)$

– Hom categories = bimodule categories

**Theorem.** $\mathcal{B} \longhookrightarrow \mathcal{B}_{\mathrm{orb}} \cong (\mathcal{B}_{\mathrm{orb}})_{\mathrm{orb}}$

Carqueville/Runkel 2012

see also Douglas/Reutter 2018, Gaiotto/Johnson-Freyd 2019

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If $X \in \mathcal{B}(\alpha, \beta)$ has *invertible* $\dim(X) \in \text{End}(1_\beta)$, then:

– $A := X^\dagger \otimes X$ is *separable* symmetric Frobenius algebra in $\mathcal{B}(\alpha, \alpha)$

– $X : (\alpha, A) \rightleftarrows (\beta, 1_\beta) : X^\dagger$ is adjoint equivalence in $\mathcal{B}_{\text{orb}}$

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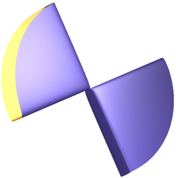
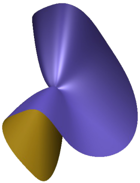
**Example.** $\mathcal{B} = \mathcal{D}_{\mathcal{Z}^{\mathrm{LG}}} \implies$  $\sim$  etc.

$\mathrm{A}_{11}$ $\qquad$ $\mathrm{E}_6$

Carqueville/Runkel 2012, Carqueville/Ros Camacho/Runkel 2013

# Orbifold defect TQFT

Let $\mathcal{Z}\colon \operatorname{Bord}^{\text{def}}_{2,1}(\mathbb{D}) \longrightarrow \mathcal{C}$ be defect TQFT.

Get new defect data $\mathbb{D}^{\text{orb}}$ with $D^{\text{orb}}_j := \big\{(2-j)\text{-cells of } (\mathcal{D}_{\mathcal{Z}})_{\text{orb}}\big\}$.

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**Definition & Theorem.**
The **orbifold defect TQFT**

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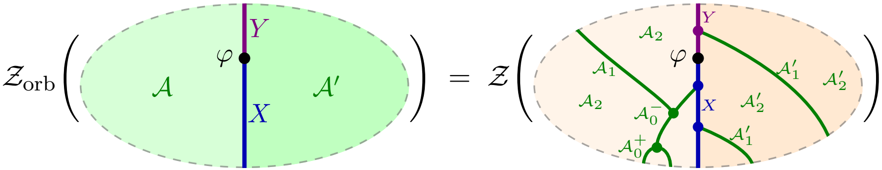
is given by substratification:




Carqueville/Runkel 2012

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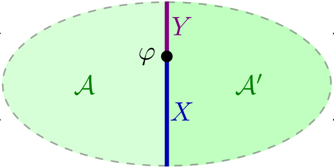


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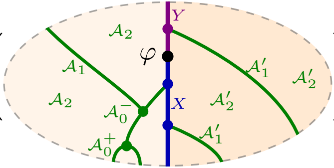



$\Bigg)$

**Example.** *Defect state sum models are Euler completed orbifolds of the trivial defect TQFT:*

$$\mathcal{Z}^{\text{ss}}_2 \;\cong\; \big(\mathcal{Z}^{\text{triv}}_2\big)^{\odot}_{\text{orb}}$$

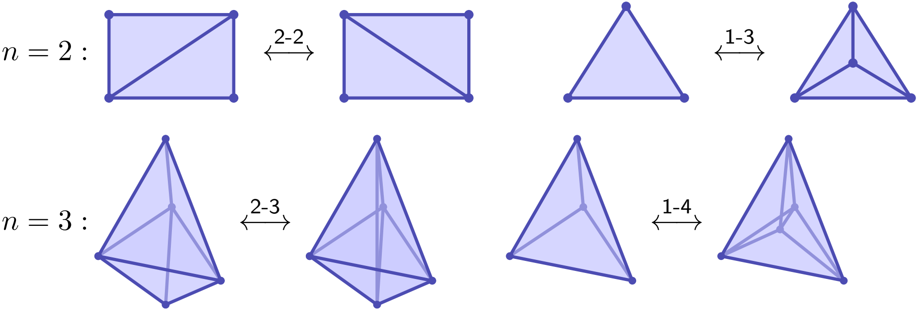
Carqueville/Runkel 2012, Lauda/Pfeiffer 2005, Davydov/Kong/Runkel 2011, Mulevičius 2022

# Orbifolds work

in any dimension $n$

# Pachner moves for $n$-dimensional triangulations

"Glue in the other side of $\partial\Delta^{n+1}$":




**Theorem.** If triangulated PL manifolds are PL isomorphic, then there exists a finite sequence of Pachner moves between them.

Pachner 1991 (for oriented variant, see Carqueville/Runkel/Schaumann 2017)

# Orbifolds in any dimension $n$

An **orbifold datum** $\mathcal{A}$ for $\mathcal{Z}\colon \mathrm{Bord}^{\mathrm{def}}_{n,n-1}(\mathbb{D}) \longrightarrow \mathcal{C}$ consists of

- $\mathcal{A}_j \in D_j$ for all $j \in \{1, \dots, n\}$,
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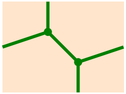
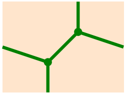
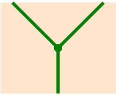
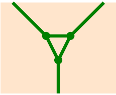
- such that (dual) "Pachner moves become identities"
    - **compatibility**:
      $\mathcal{A}_j$ is allowed decoration of $(n-j)$-simplices dual to $j$-strata
    - **triangulation invariance**:
      Let $B, B'$ be $\mathcal{A}$-decorated $n$-balls dual to two sides of a Pachner move.
      Then: $\mathcal{Z}(B) = \mathcal{Z}(B')$.

$n = 2$ is special case:

$\mathcal{Z}\Bigg($  $\Bigg) = \mathcal{Z}\Bigg($  $\Bigg)$

$\mathcal{Z}\Bigg($  $\Bigg) = \mathcal{Z}\Bigg($  $\Bigg)$

Carqueville/Runkel/Schaumann 2017

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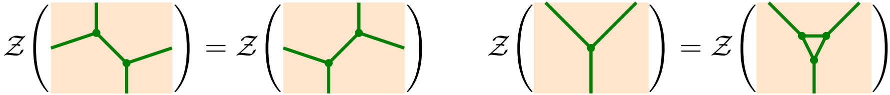

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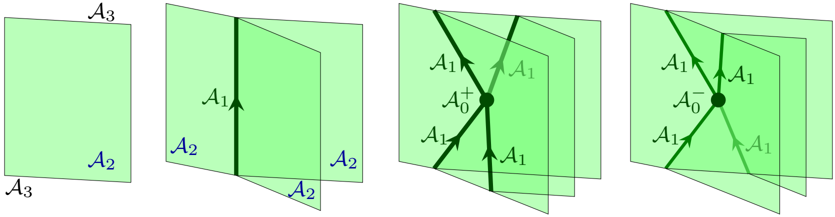
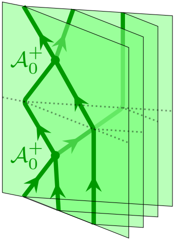
Carqueville/Runkel/Schaumann 2017

# 3-dimensional orbifold data

Let $\mathcal{T}$ be 3-category with duals. An **orbifold datum** $\mathcal{A}$ in $\mathcal{T}$ is




such that

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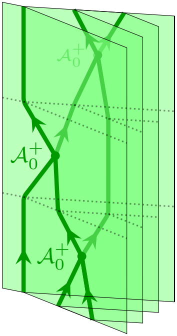
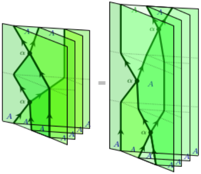
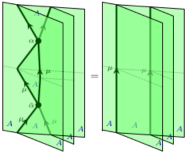



etc. (see next slide)

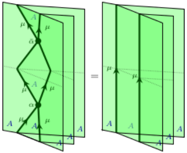
Carqueville/Runkel/Schaumann 2017



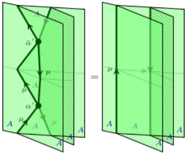
(O1)



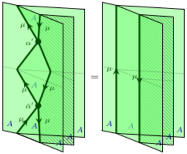
(O2)



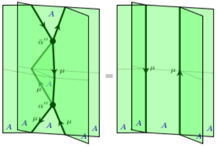
(O3)



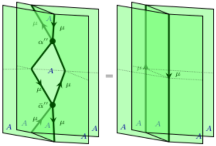
(O4)



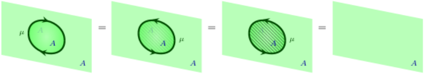
(O5)



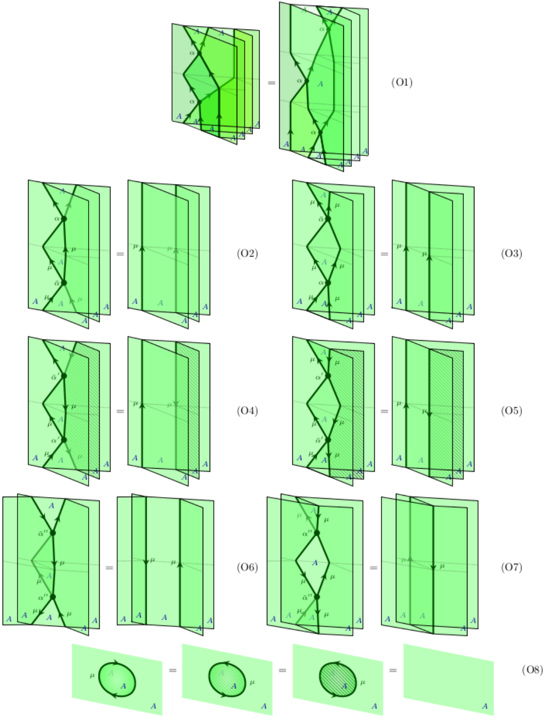
(O6)



(O7)



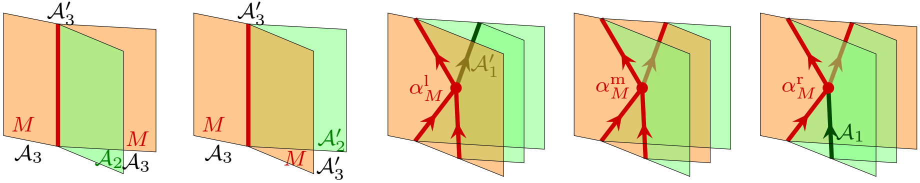
(O8)




Which 3-category $\mathcal{T}_{\mathrm{orb}}$ are orbifold data objects of?

# Representations of 3-dimensional orbifold data

Let $\mathcal{A}$ and $\mathcal{A}'$ be orbifold data in $\mathcal{T}$. An **$\mathcal{A}'$-$\mathcal{A}$-bimodule** $\mathcal{M}$ is




subject to pentagon axioms.

Johnson-Freyd/Scheimbauer 2015, Carqueville/Müller 2023

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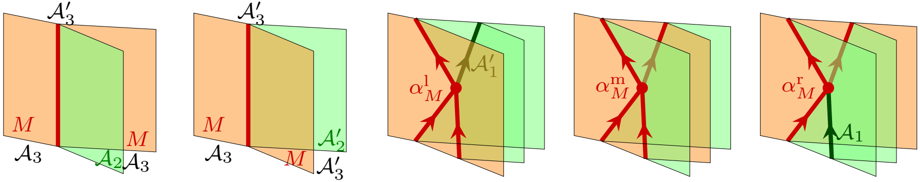
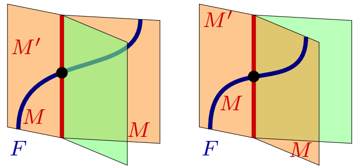



subject to pentagon axioms. A **map of $\mathcal{A}'$-$\mathcal{A}$-bimodules** $\mathcal{F}\colon \mathcal{M} \longrightarrow \mathcal{M}'$ is

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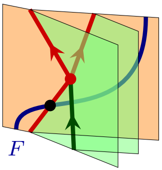


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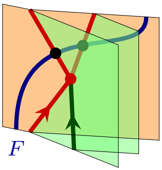
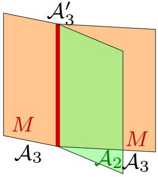


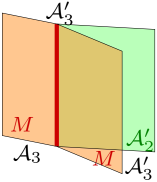

etc.

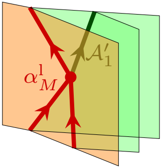
Johnson-Freyd/Scheimbauer 2015, Carqueville/Müller 2023

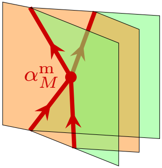
# Representations of 3-dimensional orbifold data

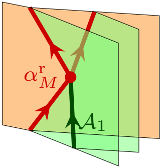
Let $\mathcal{A}$ and $\mathcal{A}'$ be orbifold data in $\mathcal{T}$. An **$\mathcal{A}'$-$\mathcal{A}$-bimodule** $\mathcal{M}$ is



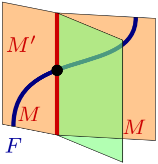


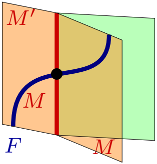




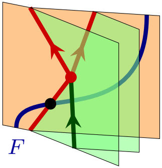



subject to pentagon axioms. A **map of $\mathcal{A}'$-$\mathcal{A}$-bimodules** $\mathcal{F}\colon \mathcal{M} \longrightarrow \mathcal{M}'$ is






such that

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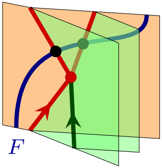
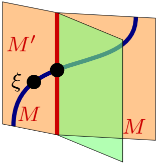



etc.

A **modification** $\xi\colon \mathcal{F} \longrightarrow \mathcal{F}'$ is $\xi\colon F \longrightarrow F'$ such that

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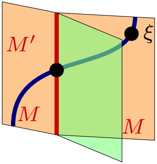


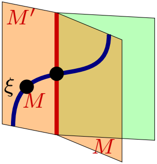


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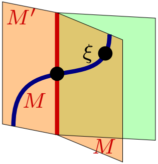



Johnson-Freyd/Scheimbauer 2015, Carqueville/Müller 2023

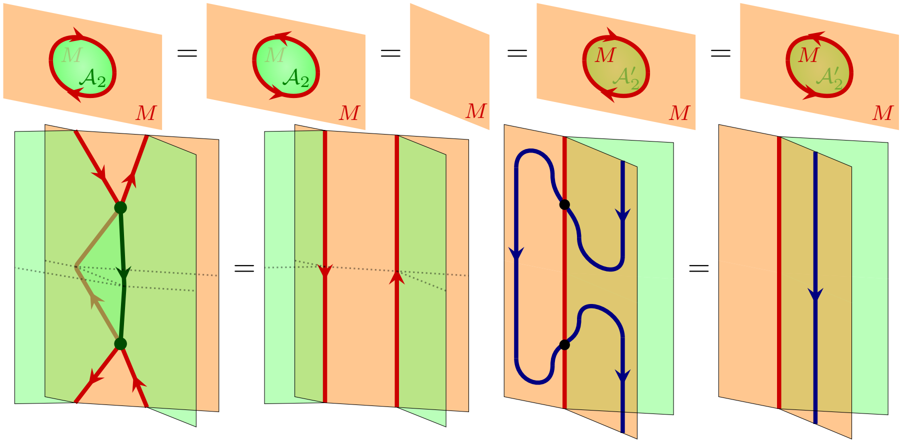
# 3-dimensional orbifold completion

(all Hom 2-categories of $\mathcal{T}$ must admit finite sifted 2-colimits that commute with composition)

The **orbifold completion** $\mathcal{T}_{\mathrm{orb}}$ of a 3-category with duals $\mathcal{T}$ has

- objects: orbifold data
- 1-cells: bimodules
- 2-cells: maps of bimodules
- 3-cells: modifications
- compositions: relative products (computed via (2-)idempotents)

such that (among other axioms)

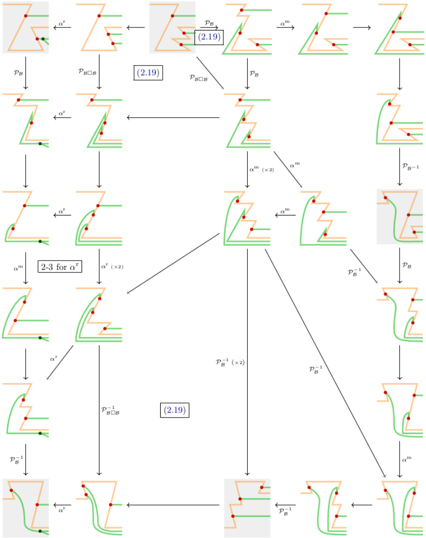



Mulevičius/Runkel 2020, Carqueville/Mulevičius/Runkel/Schaumann/Scherl 2021, Carqueville/Müller 2023

# 3-dimensional orbifold completion

**Theorem.** $\mathcal{T}_{\mathrm{orb}}$ is 3-category with adjoints for 1- and 2-cells.

*Proof:*




etc.

Carqueville/Müller 2023, Décoppet 2022

# 3-dimensional orbifold completion

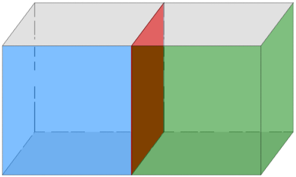
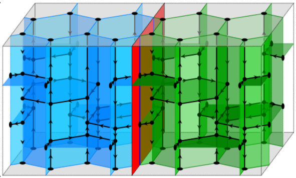
**Theorem.** $\mathcal{T}_{\mathrm{orb}}$ is 3-category with adjoints for 1- and 2-cells.

For *any* defect TQFT $\mathcal{Z}\colon \mathrm{Bord}^{\mathrm{def}}_{3,2}(\mathbb{D}) \longrightarrow \mathcal{C}$, get $\mathbb{D}^{\mathrm{orb}}$ from $(\mathcal{D}_{\mathcal{Z}})_{\mathrm{orb}}$.

**Definition & Theorem.** The **orbifold defect TQFT**

$$\mathcal{Z}_{\mathrm{orb}}\colon \mathrm{Bord}^{\mathrm{def}}_{3,2}(\mathbb{D}^{\mathrm{orb}}) \longrightarrow \mathcal{C}$$

is given by substratification:

$\mathcal{Z}_{\mathrm{orb}}\Bigg($  $\Bigg) = \mathcal{Z}\Bigg($  $\Bigg)$

Carqueville/Müller 2023 (pictures courtesy of Vincentas Mulevičius)

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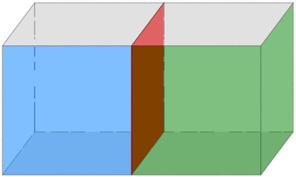

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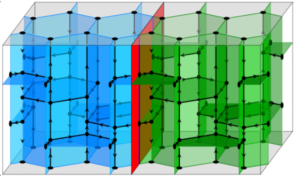

$\Bigg)$

**Example.** *Defect state sum models are Euler completed orbifolds of the trivial defect TQFT:*

$$\mathcal{Z}^{\text{ss}}_3 \cong \left(\mathcal{Z}^{\text{triv}}_3\right)^{\odot}_{\text{orb}}$$

Carqueville/Müller 2023, Kitaev/Kong 2011, Meusburger 2022

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**Theorem.** Let $\mathcal{M}$ be a modular fusion category. The 3-category in which

- objects are commutative $\Delta$-separable Frobenius algebras in $\mathcal{M}$,
- 1-cells from $B$ to $A$ are $\Delta$-separable symmetric Frobenius algebras $F$ over $(A, B)$,
- 2-cells from $F$ to $G$ are $G$-$F$-bimodules $M$ over $(A, B)$, and
- 3-cells are bimodule maps

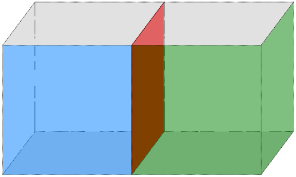
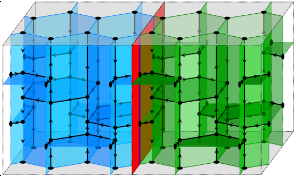
is a subcategory of $(\mathrm{B}\,\Delta\mathrm{ssFrob}(\mathcal{M}))_{\mathrm{orb}}$.

$\Longrightarrow$ recover **defect Reshetikhin–Turaev theory** à la Koppen–Mulevičius–Runkel–Schweigert

Carqueville/Müller 2023, Meusburger 2022

# Summary and outlook

**Theorem.** $\mathcal{T}_{\mathrm{orb}}$ is 3-category with adjoints for 1- and 2-cells.

$\mathcal{Z}_{\mathrm{orb}}\Bigg($  $\Bigg) = \mathcal{Z}\Bigg($  $\Bigg)$

Orbifold data are gaugeable (non-invertible) symmetries of defect TQFTs.

Carqueville/Müller 2023

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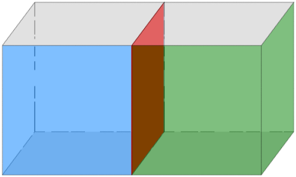
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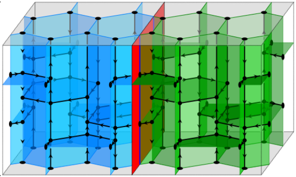
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Get **$n$-dimensional trivial defect TQFT** from $\mathrm{B}\,\mathcal{D}_{\mathcal{Z}^{\text{ss}}_{n-1}}$.

**$n$-dimensional defect state sum model** is Euler completed orbifold of the trivial defect TQFT:

$$\mathcal{Z}_n^{\text{ss}} = \left(\mathcal{Z}_n^{\text{triv}}\right)^{\odot}_{\text{orb}}$$

Recover Douglas–Reutter invariants for $n = 4$.

Carqueville/Müller 2023, Carqueville/Mulevičius/Müller 202x

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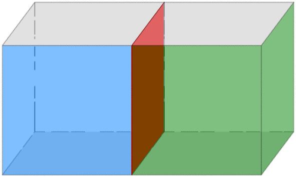
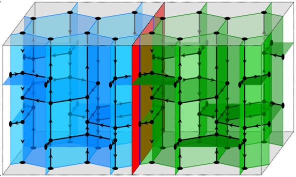
$\mathcal{Z}_{\mathrm{orb}}\Bigg($  $\Bigg) = \mathcal{Z}\Bigg($  $\Bigg)$

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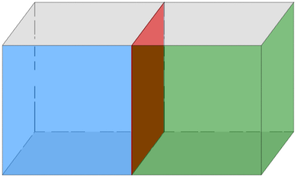
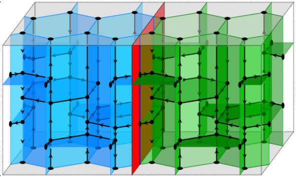
$\mathcal{Z}_{\text{orb}}\Bigg($  $\Bigg) = \mathcal{Z}\Bigg($  $\Bigg)$

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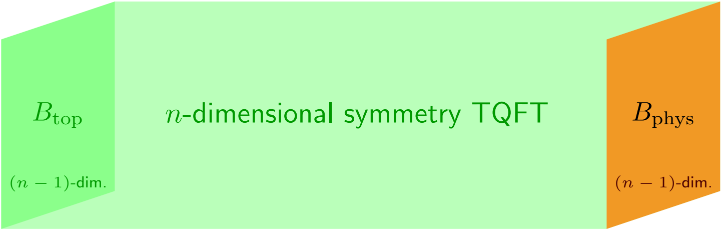
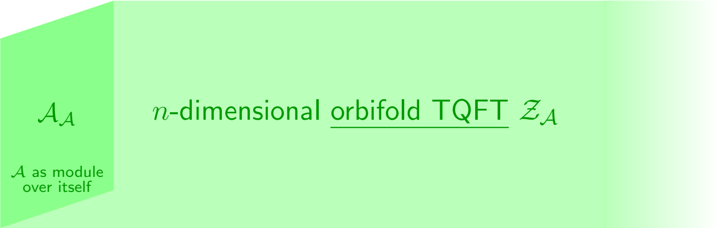
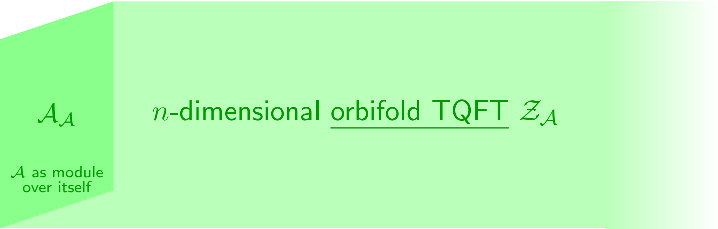
Carqueville/Müller 2023, Carqueville/Mulevičius/Müller 202x




Freed/Moore/Teleman 2022

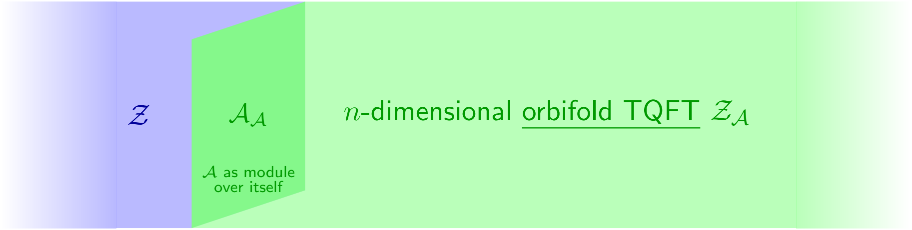



Carqueville/Runkel 2012, Carqueville/Runkel/Schaumann 2017




| $n$ | input TQFT $\mathcal{Z}$ | orbifold datum $\mathcal{A}$ | output TQFT $\mathcal{Z}_{\mathcal{A}}$ |
|---|---|---|---|
| 2 | trivial defect TQFT | $\mathbb{C}[G]$ | Dijkgraaf–Witten |
| 2 | trivial defect TQFT | *sym.* sep. Frob. $\mathbb{C}$-algebra | state sum model |
| 3 | trivial defect TQFT | $\mathrm{vect}^G$ | Dijkgraaf–Witten |
| 3 | trivial defect TQFT | *spherical* fusion category | state sum model (TVBW) |
| 3 | Reshetikhin–Turaev | many. . . | Reshetikhin–Turaev |
| 4 | trivial defect TQFT | $2\mathrm{vect}^G$ | Dijkgraaf–Witten |
| 4 | trivial defect TQFT | modular fusion category | Crane–Yetter |
| 4 | trivial defect TQFT | *spherical* fusion 2-category | state sum model (DR) |

C/Runkel/Schaumann 2012–2018, C/Mulevičius/Müller 2023, TVBW = Turaev–Viro–Barrett–Westbury, DR = Douglas–Reutter

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| 2 | trivial defect TQFT | *sym.* sep. Frob. $\mathbb{C}$-algebra | state sum model |
| 2 | Landau–Ginzburg | many. . . | non-semisimple |
| 2 | tw. sigma models | some. . . | non-semisimple |
| 3 | trivial defect TQFT | $\mathrm{vect}^G$ | Dijkgraaf–Witten |
| 3 | trivial defect TQFT | *spherical* fusion category | state sum model (TVBW) |
| 3 | Reshetikhin–Turaev | many. . . | Reshetikhin–Turaev |
| 3 | Rozansky–Witten | more work needed. . . | non-semisimple |
| 4 | trivial defect TQFT | $2\mathrm{vect}^G$ | Dijkgraaf–Witten |
| 4 | trivial defect TQFT | modular fusion category | Crane–Yetter |
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C/Runkel/Schaumann 2012–2018, C/Mulevičius/Müller 2023, TVBW = Turaev–Viro–Barrett–Westbury, DR = Douglas–Reutter