On the existence of symplectic barriers

Joint with Richard Hind and Yaron Ostrover

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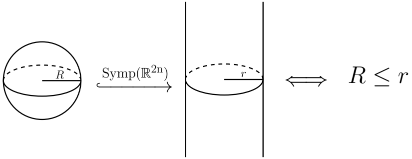

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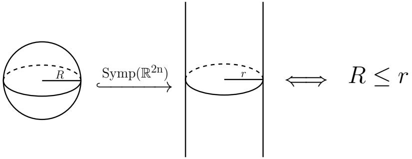

Katok – 1973: For any compact X and any $\varepsilon > 0$ there exists a symplectic map φ such that $\mathrm{Vol}(\varphi(X) \setminus Z(1)) < \varepsilon$.

Symplectic packing

Gromov – 1985: Two copies of $B^{2n}(R)$ can be symplectically embedded into $B^{2n}(1)$ such that $\varphi_1(B^{2n}(R)) \cap \varphi_2(B^{2n}(R)) = \emptyset$ if and only if $R < 1/\sqrt{2}$.

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Gromov – 1985, McDuff-PIterovich – 1994, Karshon – 1994, Traynor – 1995, Biran – 1997:

In dimension $2n = 4$:

Number of balls	2	3	4	5	6	7	8	≥ 9
Percentage of volume	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{4}{5}$	$\frac{24}{25}$	$\frac{63}{64}$	$\frac{288}{289}$	1

In dimension $2n \geq 6$:

For $2 \leq k \leq 2^n$ balls, the percentage of the volume that can be filled is $\frac{k}{2^n}$.

For $k = m^n$ for some $m \in \mathbb{N}$ there is a full packing.

Symplectic capacities

A normalized symplectic capacity on $\mathbb{R}^{2n}$ is a map c from subsets $U \subset \mathbb{R}^{2n}$ to $[0, \infty]$ with the following properties.

- If $U \subseteq V$, $c(U) \leq c(V)$,
- $c(\phi(U)) = c(U)$ for any symplectomorphism $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$,
- $c(\alpha U) = \alpha^2 c(U)$ for $\alpha > 0$,
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Gromov's width: $\underline{c}(U) = \sup\{\pi r^2 : \exists B^{2n}(r) \xhookrightarrow{\text{Symp}} U\}$

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Every normalized capacity c satisfies $\underline{c}(U) \leq c(U) \leq \bar{c}(U)$.

For ellipsoids: $c(E(a_1, a_2, \ldots, a_n)) = \min_{1 \leq i \leq n} \pi a_i^2$.

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Weinstein – 1981: "Everything is a Lagrangian submanifold".

Barriers

Barrier – Obstruction to symplectic embeddings - a submanifold the embedding must intersect.

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Brendel-Schlenk – 2022: Some Lagrangian Pinwheels in $\mathbb{CP}^2$ are barriers.

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All of these barriers are Lagrangian!

Flexibility for non Lagrangians

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Lagrangian barriers	McDuff-Polterovich – 1994: $\underline{c}(\mathbb{CP}^n \setminus \Omega) = \pi$ for Ω closed complex submanifolds.

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Abbondandolo-Schlenk – 2017:

"...The Lagrangian submanifolds are (together with energy surfaces) the most interesting submanifolds of symplectic manifolds for several reasons. One reason is that these are the submanifolds that exhibit "symplectic rigidity"..."

Theorem (H-K, Hind, Ostrover)

For every $\delta > 0$ *there exists a* **symplectic** *codimension 2 submanifold* Σ *such that*

$$\overline{c}(B^{2n}(1) \setminus \Sigma) < \delta$$

- This means that if $\varphi : B^{2n}(r) \xhookrightarrow{\text{Symp}} B^{2n}(1)$ with $\pi r^2 > \delta$, then $\varphi(B^{2n}(r)) \cap \Sigma \neq \emptyset$.

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- For the standard J, removing J-holomorphic submanifolds doesn't reduce the capacity.
- Quantifying the upper bound on $\overline{c}(B^{2n}(1) \setminus \Sigma)$ and discuss its sharpness.

Idea of the proof

$\Sigma_\varepsilon := \bigcup\{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : z_n \in \varepsilon\mathbb{Z}^2\}$ - a set of complex codim-2 hyperplanes.

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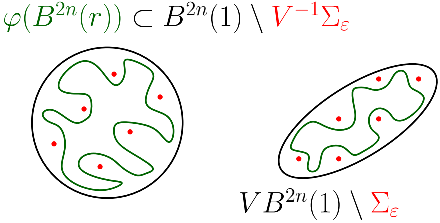

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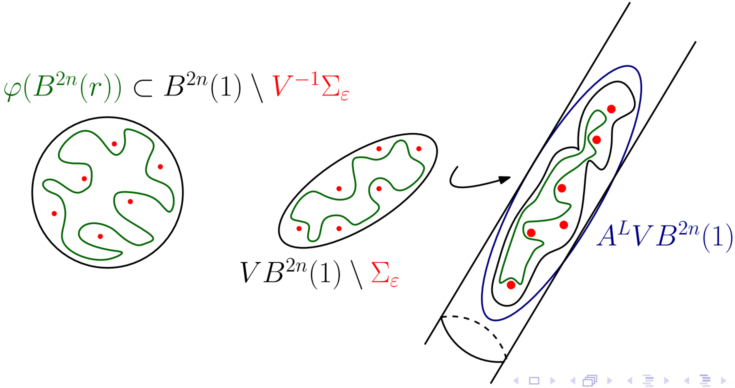

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Lemma (Eliashberg)

For every $A \in GL(2n)$ *such that* $A^*\omega \neq \lambda\omega$, *and for every* $a > 0$, *there exist* $U, V \in Sp(2n)$ *such that*

$$UAV = \left(\begin{array}{cc|c} a & 0 & \multirow{2}{*}{0} \\ 0 & a & \\ \hline \multicolumn{2}{c|}{*} & * \end{array}\right)$$

Invoking the lemma for A^L, we get that $UA^L VB^{2n}(1) \subseteq Z(a)$ and

$$c(A^L VB^{2n}(1)) \leq \pi a^2$$

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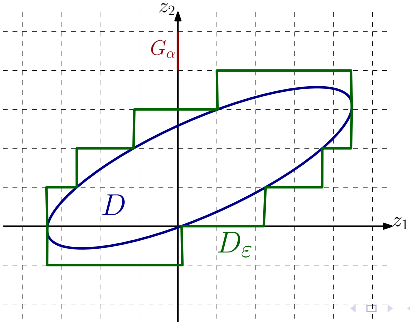
G_α - squares with vertices in $\varepsilon\mathbb{Z}^2$.

$$D_\varepsilon := \bigcup_\alpha \{(z_1, \ldots, z_{n-1}) : \exists z_n \in G_\alpha \text{ s.t. } (z_1, \ldots, z_n) \in D\} \times G_\alpha.$$



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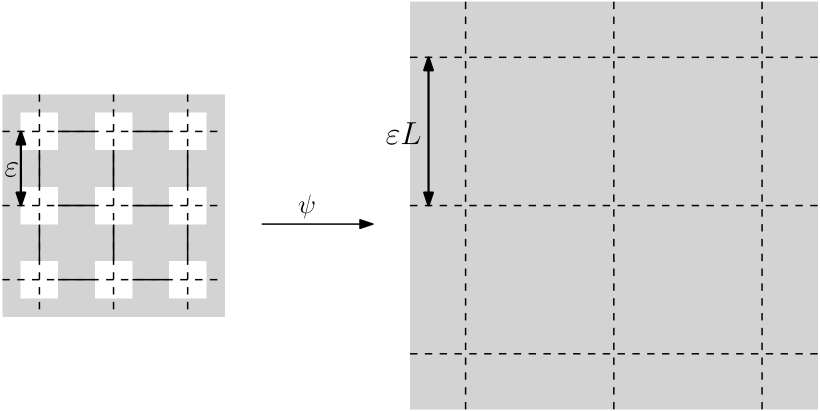
$\psi : \mathbb{R}^2 \setminus N(\varepsilon\mathbb{Z}^2) \to \mathbb{R}^2$
$Id \times \psi(D \setminus N(\Sigma_\varepsilon)) \subseteq A^L D_\varepsilon$
$\psi(G_\alpha) \subseteq L G_\alpha$
$det(\psi) = 1$



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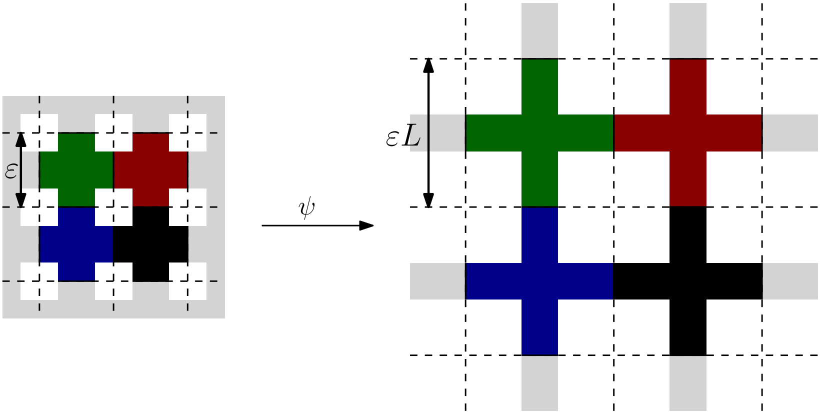

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one can try to add more and more hyperplanes to Σ in order to reduce $c(B^{2n}(1) \setminus \Sigma)$. We want to quantify the infimum of the capacity in this process.

$$g(t) := \frac{1}{\pi} \inf_{\omega(n_1,n_2)=t} \{c(B^{2n}(1) \setminus \Sigma)\}.$$

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By taking $L \to \infty$, we get $g(t) \leq t$.

Dependence on $\omega(n_1, n_2)$

Theorem (H-K, Hind, Ostrover)

$$g(t) \geq \sqrt{2\left(\frac{1}{t^2} - 1\right)\left(\sqrt{1-t^2} - 1\right) + 1} \geq t - 0.07$$

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Theorem (H-K, Hind, Ostrover)

$$\frac{1}{\pi} \inf_{\omega(n_1,n_2)=t} \{c_{HZ}(B^{2n}(1) \setminus \Sigma)\} = t.$$

Idea of the proof for the lower bounds

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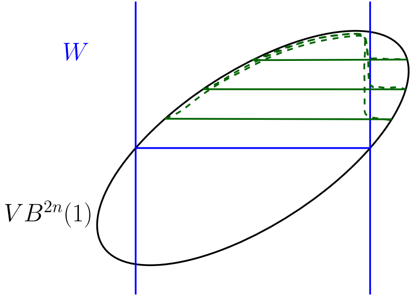
Push the hyperplanes in $V\Sigma \subset VB^{2n}(1)$ close to $\partial(VB^{2n}(1) \cap W)$ using a Hamiltonian isotopy.



Instead of investigating the complicated body $B^{2n}(1) \setminus \Sigma$, we will embed an "easier" convex body inside it.
H - subspace with Kähler angle t,
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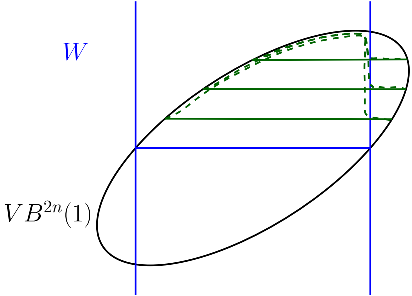

We get
Claim: $c(B^{2n}(1) \setminus \Sigma) \geq c(VB^{2n}(1) \cap W)$

Idea of the proof for the lower bounds

Gromov width of $VB^{2n}(1) \cap W$

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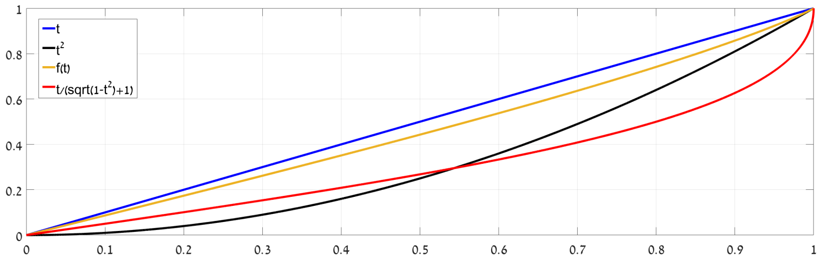
- $B^{2n}(\sqrt{f(t)}) \subseteq U(VB^{2n}(1) \cap W)$ for $f(t)$ as before.



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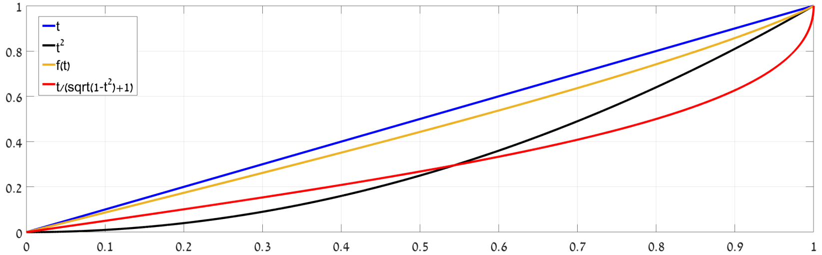

- We prove that $c_{HZ}(VB^{2n}(1) \cap W) = \pi t$.
- We conjecture that $\underline{c}(VB^{2n}(1) \cap W) = \pi t$.

$g(t) \geq t - 0.07$

Proof of the claim

1. Enough to push $\Sigma_0 \cap W$ ($\Sigma_0 := V\Sigma$) close to $\partial(VB^{2n}(1) \cap W)$ using a Hamiltonian isotopy.
2. Pushing separately along lines in $\pi_{z_n} VB^{2n}(1)$.

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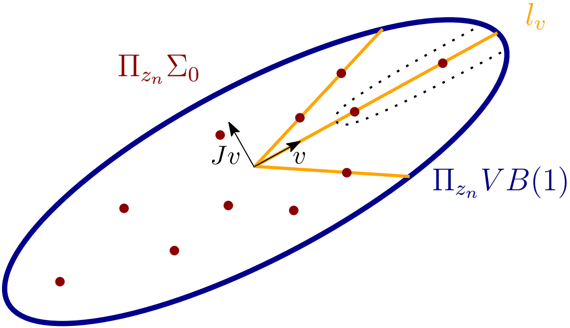

Note that for $W_v = A \oplus \{\lambda v : \lambda > \delta\}$, one has $\partial(W_v \cap VB^{2n}(1)) \subset \partial(W \cap VB^{2n}(1))$.

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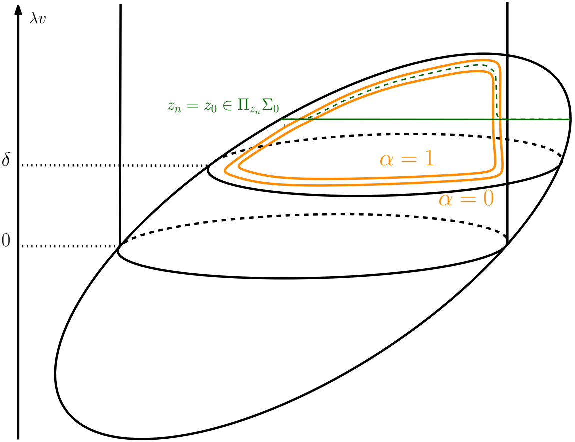

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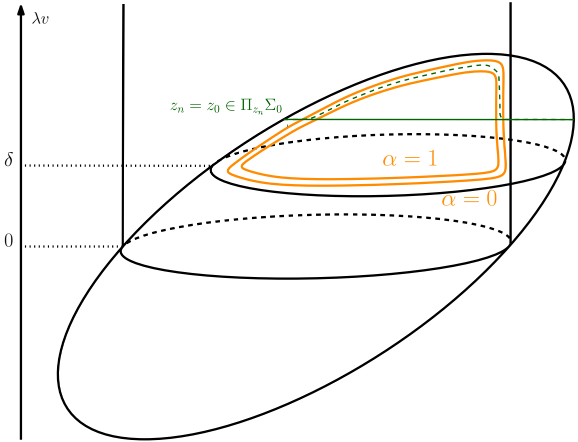

3. Define a Hamiltonian to push the hyperplanes: for $p = x + yJv$, $\langle x, Jv\rangle = 0, y \in \mathbb{R}$ put $H(p) = -\alpha(x)y$, with a cut-off outside a small nbhd of l_v.

Capacity via Hamiltonian dynamics

Ekeland, Hofer, Zehnder, Viterbo – 1989-1990: For a Hamiltonian function H and a level set c such that $\Sigma := \{x : H(x) \leq c\}$ is convex, the minimal action of a periodic Hamiltonian solution on $\partial\Sigma := \{x : H(x) = c\}$ coincides with several symplectic capacities denoted $c_{EHZ}(\Sigma)$.

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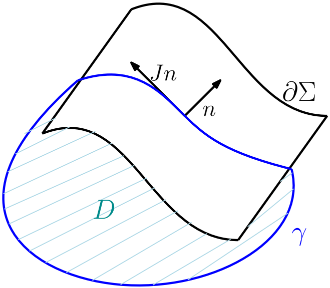

$A(\gamma) = Area_\omega(D)$

Calculating the HZ capacity

For simplicity, assume $2n = 4$. We need to prove that the action of closed characteristics on the boundary of $B^4(1) \cap V^{-1}W$ is $\geq \pi t$.

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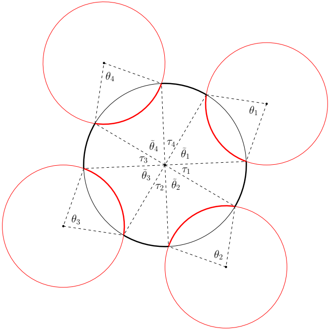
- Loops contained in $\partial V^{-1}W$ are of action πt.
- For loops that alternate between them, (after a unitary transformation) the projection to the z_2-plane is



Single hyperplane

Theorem (H-K, Hind, Ostrover)

$c(B^{2n}(1) \setminus H) = \pi\frac{1+t}{2}$ for a codimension 2 hyperplane H with $\omega(n_1, n_2) = t$, and any capacity c.

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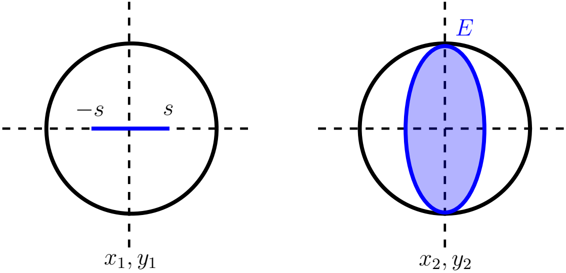
- Dynamical interpretation: removing H creates short-cuts for closed characteristics.

Idea of the proof

$H = \langle (s,0,t,0),(0,0,0,1)\rangle$, $s^2+t^2=1$. $H \cap B^4(1)$:



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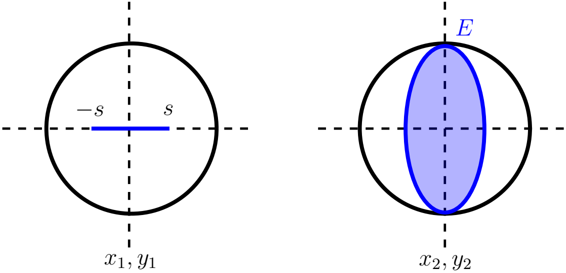

If $K \cap \{y_1 = 0\}$ doesn't intersect $\partial E \cap \{x_2 \geq 0\}$, we can displace K from the subspace H.

We take the Hamiltonian to be y_1 with a cut-off outside H.

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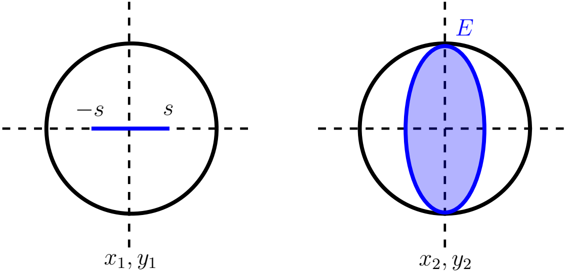

If $K \cap \{y_1 = 0\}$ doesn't intersect $\partial E \cap \{x_2 \geq 0\}$, we can displace K from the subspace H.

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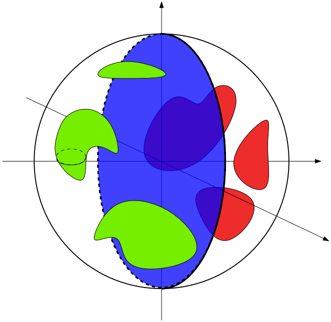
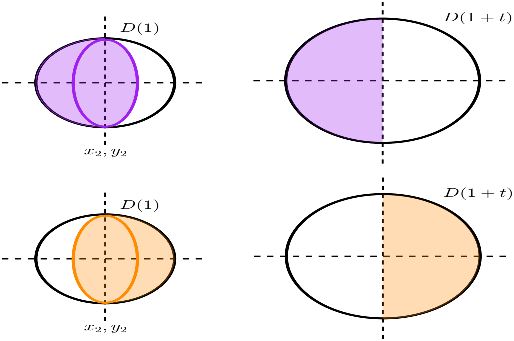
Note that the line $\partial E \cap \{x_2 \geq 0\}$ divides the disc into a region with area $\pi \frac{1+t}{2}$. We can embed $B^4(\sqrt{\frac{1+t}{2}})$ in $B^4(1) \setminus \partial E \cap \{x_2 \geq 0\}$ using a specific 2-dim area preserving map.

Single hyperplane - upper bound

Suppose that K doesn't intersect H.
In the 3-dim subspace $y_1 = 0$, H divides the space into two parts.
We use two different 2-dim area preserving maps to push the projection of both parts to x_2, y_2 away from L.





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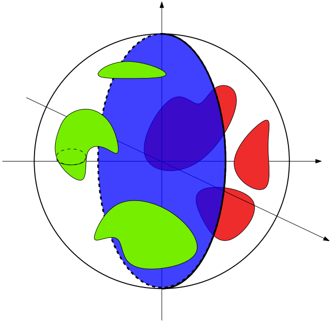

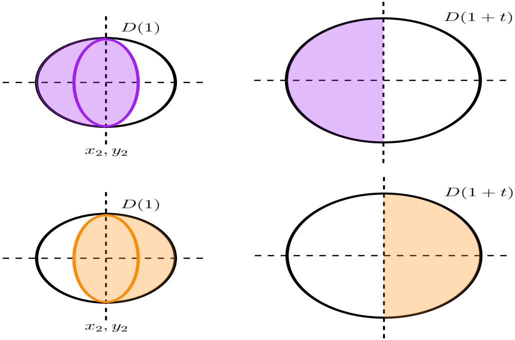

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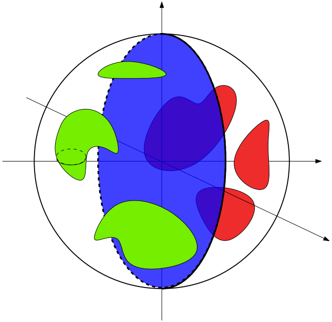

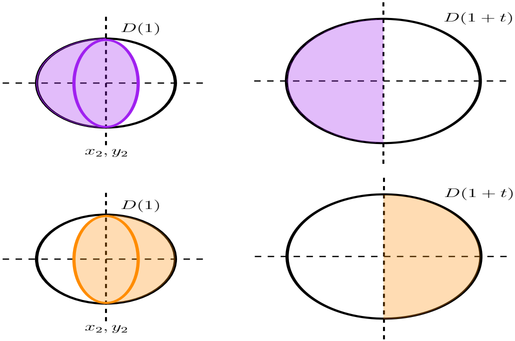

Projection of left part (purple) has area $\pi\frac{1+t}{2} \implies$ area preserving map pushing to left half disc of radius $\sqrt{1+t}$. Similarly for the right part.
Remains to define a cut-off near H between these maps without pushing other parts of K into $\{y_1 = 0\}$. For this we use the "extra room" in the x_1, y_1 disc.

Thank you!