I Classical ~~M~~ Mechanics

Newton's Mechanics:

$$\boxed{m \frac{d^2 x}{dt^2} = - \frac{\partial V}{\partial x}} \qquad (1)$$

Where $V(x,t)$ potential energy, $t \mapsto x(t)$ particle trajectory,

$\hookrightarrow E := \frac{m}{2} \dot{x}^2 + V$ total energy

Note: (1) valid only in inertial frames

Lagrangian Mechanics:

Define $\mathcal{L} := \frac{m}{2} \dot{x}^2 - V$. ("Lagrangian")

Then: (1) $\Leftrightarrow$ $\boxed{\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0}$ (2) ("Euler-Lagrange equ's")

Note: (2) holds for more general Lagrangians $\mathcal{L} = \mathcal{L}(\dot{x}, x, t)$ and does not require inertial frames ($\rightarrow$ constraints).

Hamiltonian Mechanics:

Given $\mathcal{L}$ of system.

Define $H(p,x,t) := p\dot{x} - \mathcal{L}(\dot{x},x,t)$, "Hamiltonian"

($p := \frac{\partial \mathcal{L}}{\partial \dot{x}}$, "momentum")

Then: (2) $\Leftrightarrow$ $\boxed{\begin{cases} \dot{p} = - \frac{\partial H}{\partial x} \\ \dot{x} = \frac{\partial H}{\partial p} \end{cases}}$ (3) "Hamiltonian equ of motion"

Note: If $\mathcal{L} = \frac{m}{2} \dot{x}^2 - V$, then $H = \frac{m}{2} \dot{x}^2 + V = E$.

Moreover, $\frac{d}{dt} H = \frac{\partial H}{\partial p} \dot{p} + \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial t} \overset{(3)}{=} \frac{\partial H}{\partial t}$

$\Rightarrow \frac{dH}{dt} = 0$ iff $\frac{\partial H}{\partial t} = 0$

$\hookrightarrow$ If $\frac{\partial H}{\partial t} = 0$, H is total energy.

II) Quantum Mechanics:

Given: Hamiltonian $H(p, x, t)$ of classical system.

Quantization (Dirac formalism):

(i) Assign $\left\{ \begin{array}{l} x \longrightarrow \hat{x} \\ p \longrightarrow \hat{p} \end{array} \right\}$, ~~where~~ where $\hat{p}, \hat{x}$ are operators on some Hilbert space $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ such that

$$[\hat{p}, \hat{x}] = -i.$$

(ii) Define $\hat{H} := H(\hat{p}, \hat{x}, t)$ "Hamilton operator"

$\hookrightarrow$ $\hat{H} : \mathcal{H} \longrightarrow \mathcal{H}$ shall be hermitean

(Note: ambiguities exist in def of $\hat{H}$)

(iii) ~~Time~~ Time evolution of a ~~"~~ "(quantum) state" $\Psi \in \mathcal{H}$:

$$\boxed{i \partial_t \Psi = \hat{H} \Psi} \qquad \text{"Schrödinger equ's"} \qquad (4)$$

(Note: $\Psi(t, \cdot) \in \mathcal{H}$, for any t)

Example: "Quantization in position space of $H = \frac{p^2}{2m} + V$":

- Choose $\hat{p} := -i \frac{\partial}{\partial x}$ & $\hat{x} := x$.

$\left([\hat{p}, \hat{x}](\Psi) = -i \Psi \checkmark \right)$

- ~~[illegible]~~

$\Longrightarrow$ $\hat{H} = -\frac{1}{2m} \Delta + V$

$$\hookrightarrow \boxed{i \partial_t \Psi = -\frac{1}{2m} \Delta \Psi + V \cdot \Psi} \qquad \text{"Schrödinger equ"} \qquad (5)$$

Choose $\mathcal{H}$ as a subspace of $L^2(\mathbb{R}^3)$ & $\langle \cdot | \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^3)}$.

Postulates (Quantum Measurements):

1) Measuring equipment $\longleftrightarrow$ Hermitean operator $A: \mathcal{H} \to \mathcal{H}$.

2) Measurement on a (prepared) state Ψ is the assignment of Ψ to some eigenstate Ψ_λ of A, (i.e. $A\Psi_\lambda = \lambda \Psi_\lambda$). ($\to$ "Collapse of Wave function")

3) The outcome of measurement is the eigenvalue λ of A, (corresponding to Ψ_λ in (2)).

4) Propability of measuring λ is $\langle \Psi_\lambda | \Psi \rangle_{\mathcal{H}}^2$.

Remark:

- Postulate (4) requires "normalization": $\boxed{\|\Psi\|_{\mathcal{H}} = 1}$

  ($\to \forall c \in \mathbb{C}$, Ψ & $c\Psi$ are physically indistinguishable)

- Quantum Mechanics allows for discrete state spaces,
  ($\longrightarrow$ Bohr's model of atom)

Uncertainty Principle (Heisenberg):

Thm: Let A, B be hermitean operator,

$$\Rightarrow \langle \Psi | A | \Psi \rangle \langle \Psi | B | \Psi \rangle \geq \frac{1}{2} \left| \langle \Psi | [A, B] | \Psi \rangle \right|$$

Example:

$$\Delta \hat{p} \, \Delta \hat{x} \geq \frac{\hbar}{2}$$

(Note: We used $\hbar = 1$ above.)

I) Relativistic Quantum Mechanics

Goal: Quantize relativistic energy $E^2 = p^2 + m^2$! (set $V \equiv 0$)

↳ Use $\hat{p} := -i\frac{\partial}{\partial x}$, $\hat{x} := x$

$$\Rightarrow \hat{H}^2 = -\Delta + m^2$$

Convention:

$$\eta = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

Minkowski metric

1) Klein-Gordon-eqn:

Use Schrödinger eqn, $i\partial_t \Psi = \hat{H}\Psi$, to get

$$i\partial_t(i\partial_t\Psi) = \hat{H}^2\Psi = -\Delta\Psi + m^2\Psi$$

$$\Leftrightarrow \underbrace{\partial_t^2\Psi - \Delta\Psi}_{=:\Box\Psi} + m^2\Psi = 0$$

$$\Leftrightarrow \boxed{(\Box + m^2)\Psi = 0}$$

(KGE) "Klein-Gordon eqn"

Rmk:
(KGE) describes particles with integer spin.

2) Dirac equation:

Motivation: (KGE) is 2^{nd}-order and might not contain full information of underlying "formal" Schrödinger eqn

↳ Want: $i\partial_t\Psi = \hat{H}\Psi$

for "$\hat{H} = \sqrt{\underbrace{-\Delta}_{=\hat{p}^2} + m^2}$"

~~Ansatz:~~ $\cancel{\hat{H} = i\gamma^0\gamma^j\partial_j - \gamma^0 m \quad (j=1,2,3)}$

Ansatz: $\hat{H} = i\gamma^0\gamma^\alpha \frac{\partial}{\partial x^\alpha} - \gamma^0 m$, $(\alpha = 1,2,3)$

where γ^μ $(\mu = 0,\dots,3)$ are 4×4-matrices over $\mathbb{C}$ satisfying: (i) γ^0 & $\gamma^0\gamma^\alpha$ are hermitean $(\langle\cdot,\cdot\rangle_{\mathbb{C}^4})$

(ii) $\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}_4$

("Clifford algebra")

$\Rightarrow$ $\hat{H}$ is hermitean ✓

& $\hat{H}^2\Psi = -\Delta\Psi + m^2\Psi$ ✓

~~$i\partial_t\Psi = i\gamma^0(\gamma^\alpha\partial_\alpha - m)\Psi$ "Dirac eqn"~~

$\Rightarrow$ $i\partial_t\Psi = \hat{H}\Psi = i\gamma^0(\gamma^\alpha\partial_\alpha - m)\Psi$

for time evolution of Ψ

$\overset{(\gamma^0)^2 = 1}{\Leftrightarrow}$ $\boxed{(i\gamma^\mu\partial_\mu - m)\Psi = 0}$ (DE) "Dirac equation"

Remarks:

- $\Psi(x) = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}(x) \in \mathbb{C}^4$, contains info about spin (up/down) and positive/negative energy ~~particle/antiparticle~~

  $\hookrightarrow$ (DE) describes particles/anti-particles of spin $1/2$

- (DE) is symmetric hyperbolic PDE ~~[illegible]~~ (Kato et. al.)

"Dirac representation" of γ^μ:

$$\gamma_D^0 := \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad \gamma_D^\alpha := \begin{pmatrix} 0 & \sigma^\alpha \\ -\sigma^\alpha & 0 \end{pmatrix}$$

for $\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. ("Pauli matrices")

Framework:

- $\langle \psi | \phi \rangle(x) := \sum_{j=1}^{4} s_j \underbrace{\psi^j(x)^*}_{\text{complex conjugate}} \phi^j(x)$, $s_1 = 1 = s_2$ & $s_3 = -1 = s_4$ ("Spin-Scalar-product")

  $\hookrightarrow$ γ_D^μ hermitean w.r.t $\langle\cdot|\cdot\rangle$: $\left(\langle \gamma^\mu \psi | \phi \rangle = \langle \psi | \gamma^\mu \phi \rangle \right)$

- $J^\mu := \langle \psi | \gamma^\mu \psi \rangle$ "Dirac current"

  $\hookrightarrow$ If ψ solves (DE), then $\partial_\mu J^\mu = 0$ (current conservation)

- $(\psi|\phi) := \int_{\mathbb{R}^3} \langle \psi | \gamma^0 \phi \rangle (t, \vec{x})\, d^3\vec{x}$

  defines scalar-product on $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$.

  [Note: $(\psi|\phi)$ time-indep. by current conservation:

  $$\int_{\mathbb{R}^3} \langle \psi | \gamma^0 \phi \rangle (t_1, \cdot)\, d^3\vec{x} - \int_{\mathbb{R}^3} \langle \psi | \gamma^0 \phi \rangle (t_2, \cdot)\, d^3\vec{x} = \int_{t_2}^{t_1} \int_{\mathbb{R}^3} \overbrace{\partial_\mu J^\mu}^{=0}\, d^4x = 0.$$
  ]

Lorentz invariance:

Consider $x^\mu \longrightarrow \Lambda^\mu{}_\nu \tilde{x}^\nu$, Lorentz transfo (orthochronous, proper).

Then $\exists\, U(\Lambda)$ unitary matrix with respect to $\langle\cdot|\cdot\rangle(x)$

such that $(\Lambda^{-1})^\mu{}_\nu \gamma^\nu = U^{-1}(\Lambda) \gamma^\mu U(\Lambda)$.

Thus, if ψ solves $(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m)\psi = 0$, then $(i\tilde{\gamma}^\mu \frac{\partial}{\partial \tilde{x}^\mu} - m)(U\psi) = 0$

$\Rightarrow$ Lorentz transfo induces unitary transfo $\psi \longrightarrow U\psi$

III. Gauge Symmetries in Minkowski space

- Let G be a Lie group representation on $\mathbb{C}^4 \times \mathbb{R}^{1,3}$.

  ($U(x)$ invertible 4×4 matrix $\forall x \in \mathbb{R}^{1,3}$, $U \in G$.)

- "Gauge transformation": $\Psi \longrightarrow \Psi' := U\Psi$ ←

  Motivation: Some gauge transfos leave physical state unchange, say, by preserving propability.

  ↳ Problem: If $(i\gamma^\mu \partial_\mu - m)\Psi = 0$,

  then $(i\gamma^\mu\partial_\mu - m)\Psi' = (i\gamma^\mu \partial_\mu U)\Psi \underset{\text{e.g.}}{\neq} 0$

  Resolution: Introduce "gauge potential" $A_\mu \in \mathfrak{g}$,

  ↑ Lie algebra of G

  which gauge transforms as

  $$A_\mu \longrightarrow A'_\mu := U A_\mu U^{-1} + i U \partial_\mu U^{-1}$$

  and replace ∂_μ by $\boxed{D_\mu := \partial_\mu - iA_\mu}$

  "Spin connection"

- Notation: $\not{D} := i\gamma^\mu D_\mu$ "Dirac operator"

Thm:

Assume Ψ solves $(\not{D} - m)\Psi = 0$,

Then $\Psi' := U\Psi$ solves $(\not{D}' - m)\Psi' = 0$,

where $\not{D}' := i\gamma'^\mu D'_\mu$ for $\gamma'^\mu = U\gamma^\mu U^{-1}$ & $D'_\mu = \partial_\mu - iA'_\mu$,

& $A'_\mu = UA_\mu U^{-1} + iU\partial_\mu U^{-1}$.

Moreover, γ'^μ satisfies Clifford algebra.

proof:

- γ'^μ satisfies Clifford algebra, since

  $$\{\gamma'^\mu, \gamma'^\nu\} = U\{\gamma^\mu, \gamma^\nu\}U^{-1} = U(2\eta^{\mu\nu})U^{-1} = 2\eta^{\mu\nu}. \checkmark$$

• We now show that $\not{D}'(U\psi) = U\not{D}\psi$, which then implies $(\not{D}' - m)(U\psi) = U\underbrace{(\not{D} - m)(\psi)}_{=0} = 0$. ✓

$$\hookrightarrow D'_\mu(U\psi) = (\partial_\mu - iA'_\mu)(U\psi)$$

$$= U\partial_\mu\psi + (\partial_\mu U)\cdot\psi - i\underbrace{A'_\mu U\psi}_{= UA_\mu\psi + iU(\partial_\mu U^{-1})U\psi}$$

$$= U\underbrace{(\partial_\mu - iA_\mu)}_{= D_\mu}\psi + (\partial_\mu U)\cdot\psi + U\underbrace{\partial_\mu(U^{-1})}_{= -U^{-1}\partial_\mu U\cdot U^{-1}}U\psi$$

$$= UD_\mu\psi$$

$$\Rightarrow \not{D}'(U\psi) = i\gamma'^\mu D'_\mu(U\psi)$$

$$= iU\gamma^\mu \cancel{U^{-1}}\cdot\cancel{U}D_\mu\psi$$

$$= U\not{D}\psi. ✓$$

Example: U(1) gauge symmetry of electrodynamics:

Let $G = U(1)$, that is, $G = \{e^{i\phi(x)} \mid \phi(x) \in \mathbb{R}\}$.

Then $\not{D} = i\gamma^\mu(\partial_\mu - ia_\mu)$, $a_\mu \in \mathbb{R}$, $(\mu = 0, \dots, 3)$

& $a_\mu \longrightarrow a'_\mu := a_\mu + i\partial_\mu\phi$

which is precisely the gauge transformation of the ~~electromagnetic potential~~ vector-potential in classical electrodynamics. ✓

Note: $\langle (e^{i\phi}\psi) | e^{i\phi}\psi\rangle = \langle\psi|\psi\rangle$, U(1) preserves spin scalar product.

V Gravity as a local SU(2,2) gauge symmetry (Finster, '99)

- Let $(\mathcal{M}, g)$ be 4-D Lorentz manifold.
  ↳ We restrict consideration to coordinate charts (x, Ω), to avoid global issues (like topological restrictions to existence of spinors).
- Locally, we can take "spinor bundle" $S\mathcal{M}$ as $S\mathcal{M} = \mathbb{C}^4 \times \Omega$.
  ↳ Spinors Ψ are sections in $S\mathcal{M}$, (i.e. $\Psi(x) \in \mathbb{C}^4 \; \forall x \in \Omega$).
- Choose on sections in $S\mathcal{M}$ (pointwise) an inner product $\langle \cdot | \cdot \rangle$ of signature $(2,-2)$. $\langle \cdot | \cdot \rangle$ is called "spin scalar product".
- Generalize Clifford algebra, $(\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{1}_4)$, to $\mathcal{M}$:

$$\{G^\mu, G^\nu\} = 2 g^{\mu\nu} \mathbb{1}_4$$

4×4 matrices, $(\mu, \nu \in 0, \dots, 3)$

- Choose G^μ hermitean w.r.t $\langle \cdot | \cdot \rangle$ & such that $\forall p \in \mathcal{M}$ $\exists$ coord's x^μ & $\exists$ gauge : $\boxed{G^\mu(p) = \gamma^\mu_0}$.
  ↳ <u>Ansatz</u>:

$$\mathcal{D} := i\, G^\mu \left(\frac{\partial}{\partial x^\mu} - i A_\mu \right)$$

<u>"Dirac operator"</u>

$A_\mu|_x \in \mathbb{C}^{4\times4}$, $\forall \mu = 0, \dots, 3$

- <u>Goal</u>: Derive $A_\mu|_x \in \mathbb{C}^{4\times4}$ as "gauge potential" corresponding to "gauge group"

$$U(2,2) := \left\{ x \longmapsto U(x) \in GL(\mathbb{C}^4) \,\middle|\, \langle U\Psi | U\phi \rangle(x) = \langle \Psi | \phi \rangle(x) \;\forall \Psi, \phi \text{ spinors} \right\}$$

the point-wise unitary matrices w.r.t. $\langle \cdot | \cdot \rangle$!

↳ Find A_μ which transforms as

$$A_\mu \longrightarrow A'_\mu := U A_\mu U^{-1} + U \partial_\mu U^{-1} \quad \forall U \in U(2,2)$$

Thm:

$$A_\mu = E_\mu + a_\mu$$

where $E_\mu := \frac{i}{2}\rho(\partial_\mu \rho) - \frac{i}{16}\mathrm{Tr}(G^\rho \nabla_\mu G^\sigma) G_\rho G_\sigma + \frac{i}{8}\mathrm{Tr}(\rho G_\mu \nabla_\sigma G^\sigma)\rho$

& $\rho := \frac{i}{4!}\sqrt{|\det g|}\,\epsilon_{j k l m}\, G^j G^k G^l G^m$

& $G_\rho := g_{\rho\sigma} G^\sigma$

and a_μ ~~...~~ is a scalar, transforming as $a_\mu \to a_\mu + U\partial_\mu U^{-1}$. ∎

proof: See [Finster, '98].

Remark:

(i) a_μ can be interpreted as vector potential of electrodynamics.
"$\Rightarrow$" E_μ decribes gravitational field

(ii) E_μ is trace-free, $\Rightarrow$ E_μ is in Lie algebra of $SU(2,2)$
$\hookrightarrow$ $SU(2,2)$ gauge group of gravity

(iii) If $g = \eta$, $\Rightarrow$ $E_\mu = 0$.

Why choosing $U(2,2)$ to describe gravity?

- Lorentz transformations induce (global) $U(2,2)$ transformations on Ψ. (soln of Dirac-eqn.)
- $U(2,2)$ is largest group that preserves spin scalar product locally.

~~Remark:~~ **Lemma:**

Let $N \subset M$ be a Cauchy surface. Then $(\Psi|\phi)_N := \int_N \langle \Psi(x) | G^\mu \nu_\mu \phi(x)\rangle \, d\mu_N(x)$

is positive definite and independent of N.

proof: Similar to Minkowski-case. ∎

$\to (*)$: Since E_μ is only comprised of G^μ, a coord change to G^α appears again in E_μ
$\Rightarrow$ E_μ decribes gravity ✓

Remark:

Under coord transformation $x^\mu \to y^\alpha$, transform Dirac matrices

as $G^\mu \to G^\alpha := \frac{\partial y^\alpha}{\partial x^\mu} G^\mu$.

Then E_μ transforms as a tensor and $\mathcal{D}$ & Ψ transforms as a scalar,

$$\mathcal{D} = i G^\mu\left(\frac{\partial}{\partial x^\mu} - i A_\mu\right) = i \underbrace{G^\mu \frac{\partial y^\alpha}{\partial x^\mu}}_{= G^\alpha}\left(\frac{\partial}{\partial y^\alpha} - i A_\alpha\right).$$

$(*) \to$

$\to$ Solutions Ψ are mapped to solutions.