Traces and higher structures

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Based on work with
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May 22, 2024

"Lisbon", 22. 5. 2024

Traces and higher structures

Overview:

Traces on (linear / monoidal) categories $\longrightarrow$ Extended graphs $\longleftarrow$ TFT with defects

Guiding question: Categorical dimensions for traces?

Traces and higher structures

Overview:

$$\begin{tikzcd} \text{Traces on } \begin{pmatrix}\text{linear}\\ \text{monoidal}\end{pmatrix} \text{ categories} \arrow[dr] & & \text{TFT with defects} \arrow[dl] \\ & \text{Extended graphs} & \end{tikzcd}$$

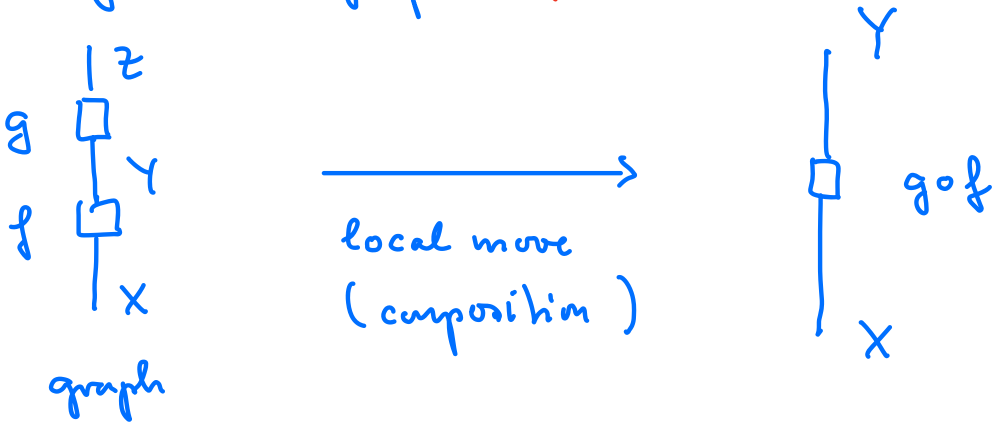
Guiding question: Categorical dimensions for traces?

Summary:

1. Graphical calculus
2. Traces and Nakayama functors
3. State-sum models
4. Extended graphs and their graphical calculus

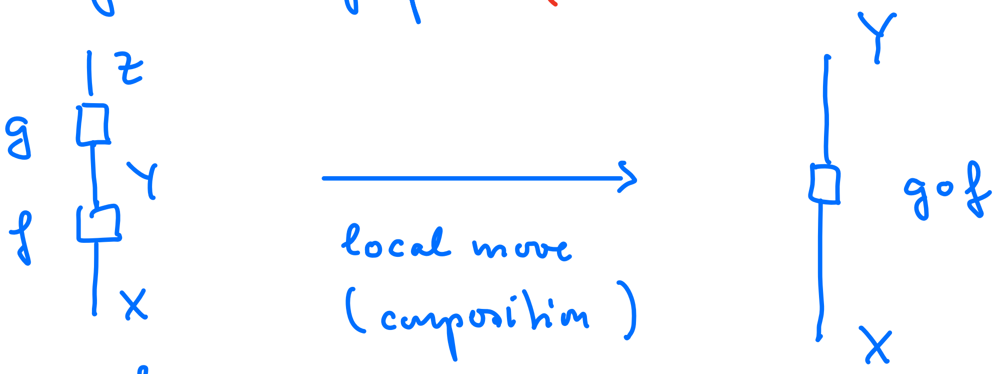
1. Graphical calculus

1.1 Categories and graphs (one-dim)



1. Graphical calculus

1.1 Categories and graphs (one-dim)



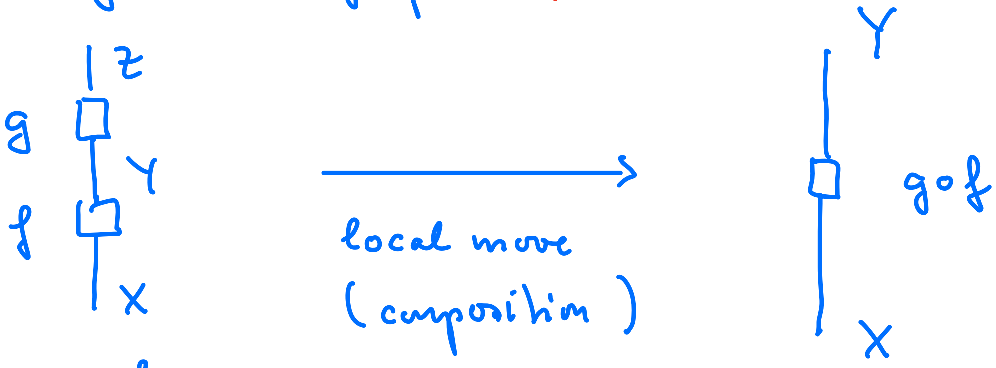
graph

Two aspects

- geometric move
- composition map on morphisms

1. Graphical calculus

1.1 Categories and graphs (one-dim)



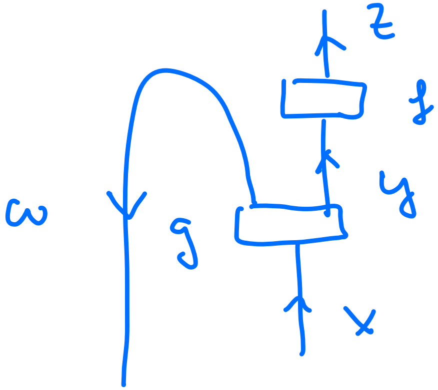
graph

1.2 Monoidal categories (two-dim)

$(W \otimes f) \circ g : X \to W \otimes Z$

(two compositions,
$\otimes$ and $\circ$)

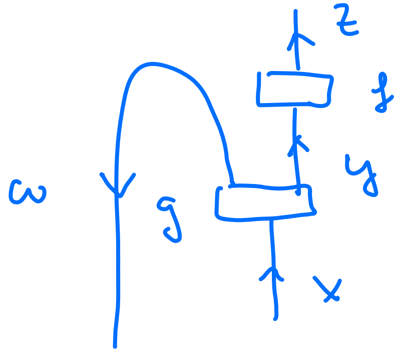
1.3 Rigid monoidal categories : left and right duals



$\rightsquigarrow$ $f \circ (ev_w \otimes y) \circ (w^\vee \otimes g)$:

$w^\vee \otimes X \longrightarrow Z$

1.3 Rigid monoidal categories : left and right duals



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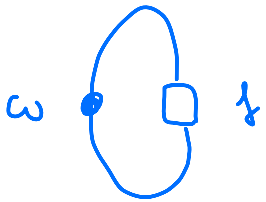
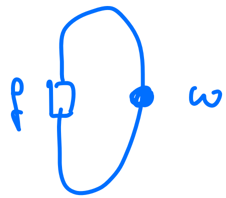
1.4. Pivotal monoidal categories $\quad \omega : id \xrightarrow[\otimes]{\sim} ?^{\vee\vee}$



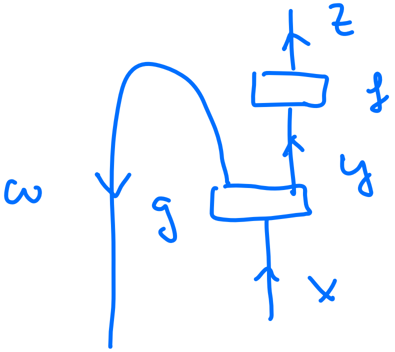
$= tr^l(f)$

$tr^r(f) =$



two traces

1.3 Rigid monoidal categories : left and right duals



$\rightsquigarrow$ $f \circ (\mathrm{ev}_\omega \otimes y) \circ (\omega^\vee \otimes g) :$

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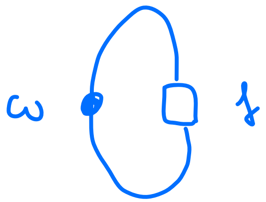
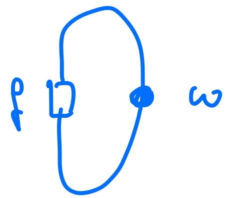


$= \mathrm{tr}^l(f)$

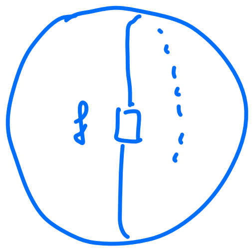
$\mathrm{tr}^r(f) =$



two traces

1.5 (Trace) spherical : both traces coincide

$\rightsquigarrow$ evaluation on sphere



Are traces best considered in 2d setting?

2. Traces and the Nakayama functor

(In linear algebra, trading traces w/o tensor product. Bases used.)

Now, let $\mathcal{A}$ be a finite linear category

2.1 Nakayama functor

$$N^r_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$$

$$N^r_{\mathcal{A}}(-) = \int^{a \in \mathcal{A}} \mathrm{Hom}(-, a)^* \otimes a \quad \in \mathrm{Rex}(\mathcal{A}, \mathcal{A})$$

$$\begin{tikzcd}
 & \mathcal{A} \boxtimes \mathcal{A}^{op} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\mathrm{Rex}(\mathcal{A}, \mathcal{A}) \arrow[rr, leftrightarrow, "\sim"] & & \mathrm{Lex}(\mathcal{A}, \mathcal{A})
\end{tikzcd}$$

$N^r \mapsto \mathrm{id}$, $\mathrm{id} \to N^l$

$\mathrm{id} \leftarrow \mathrm{id}$

2.1 Nakayama functor

$$N^r_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$$

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<u>Remarks</u>

- $\mathcal{A} = A\text{-mod}_{f.d.}$, A f.d. k-algebra

  Right exact $\to$ Eilenberg-Watts thm implies

  $$N^r_{\mathcal{A}}(-) \cong N^r(A) \otimes_A -$$

- $$N^r_{\mathcal{A}}(A) = \int^{y \in A\text{-mod}} \mathrm{Hom}_A(A, y)^* \otimes y \cong \int^{y \in A\text{-mod}} y^* \otimes y \cong A^*$$

  Peter-Weyl

2.2. Thm [FSS] [Main thm on Nakayama functor]

$\mathcal{A}, \mathcal{B}$ finite, $F \in \mathcal{L}ex(\mathcal{A}, \mathcal{B})$, F^{la} left exact and F^{lla} left exact.

Then

$$N^{l}_{\mathcal{B}} \circ F \cong F^{lla} \circ N^{l}_{\mathcal{A}}$$

coherent with composition.

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coherent with composition.

2.3 "Modified" traces on finite categories

A first calculation: $\quad P$ projective:

$$\mathrm{Hom}_{\mathcal{A}}\big(P, N^r_{\mathcal{A}}(X)\big) = \int^{Y \in \mathcal{A}} \mathrm{Hom}\big(P, \mathrm{Hom}(X,Y)^* Y\big)$$

$$= \int^{Y \in \mathcal{A}} \mathrm{Hom}(X,Y)^* \otimes \mathrm{Hom}(P,Y) \underset{\text{Yoneda}}{\cong} \mathrm{Hom}(X,P)^*$$

2.3 "Modified" traces on finite categories

$$\mathrm{Hom}_{\mathcal{A}}(p, N^r_{\mathcal{A}}(x)) \cong \mathrm{Hom}(x, p)^*$$

Bilinear form (p, q projective):

$$\langle \cdot , \cdot \rangle : \mathrm{Hom}_{\mathcal{A}}(p,q) \otimes \mathrm{Hom}(p, N^r(p)) \cong \mathrm{Hom}(q, N^r(p))^* \otimes \mathrm{Hom}(q, N^r(p)) \xrightarrow[\text{non deg}]{} \Bbbk$$

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$$\mathrm{Hom}_{\mathcal{A}}(p, N_{\mathcal{A}}^{r}(x)) \cong \mathrm{Hom}(x,p)^{*}$$

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<u>Def</u>:

$$t_p : \mathrm{Hom}(p, N^{r}(p)) \longrightarrow \Bbbk \qquad \text{modified trace}$$
$$f \longmapsto \langle id, f \rangle$$

2.3 "Modified" traces on finite categories

$$\mathrm{Hom}_{\mathcal{A}}(p, N^r_{\mathcal{A}}(x)) \cong \mathrm{Hom}(x,p)^*$$

Def: $t_p : \mathrm{Hom}(p, N^r(p)) \longrightarrow k$ modified trace

$$f \longmapsto \langle id, f \rangle$$

Prop [SW]

1. Cyclic: $t_q(q \xrightarrow{g} p \xrightarrow{f} N^r q) = t_q(f \circ g)$

$$= t_p(p \xrightarrow[f]{} N^r q \xrightarrow[N^r g]{} N^r p) = t_p(N^r g \circ f)$$

2. Non-deg.

Q: Are traces best considered in a 1d setting?

2.4. Modified traces on finite tensor categories

<u>Thm</u> [FSS] (Radford S^4)

$\mathcal{A}$, $\mathcal{B}$ finite tensor categories, $\mathcal{M}$ an $\mathcal{A}$-$\mathcal{B}$ bimodule

$$N^{\ell}_{\mathcal{M}}(a.m.b) \cong a^{\vee\vee}.N^{\ell}_{\mathcal{M}}(m).{}^{\vee\vee}b$$

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Thm [FSS] (Radford S^4)

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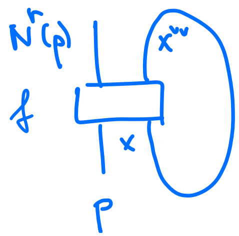
$$N^l_{\mathcal{M}}(a.m.b) \cong a^{\vee\vee}.N^l_{\mathcal{M}}(m).{}^{\vee\vee}b$$

Def. [partial trace]

$$tr^x_r : \mathrm{Hom}(p\,x, N^r(p\,x)) \cong \mathrm{Hom}(p\,x, N^r(p)\,x^{\vee\vee})$$

$$\to \mathrm{Hom}(p, N^r(p))$$



no pivotal structure on $\mathcal{C}$ needed

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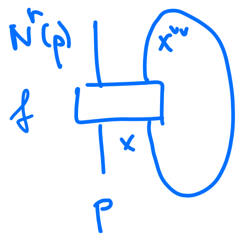


no pivotal structure on $\mathcal{C}$ needed

Prop [SW] $\quad t_{px}(f) = t_p(tr^x_r\, f) \quad$ partial trace property

Eilenberg-Watts

balanced Deligne product

<u>Note</u> : $N^{\mu}_{\mathcal{M}} \in \mathcal{R}ex^{tw}_{A|B}(\mathcal{M},\mathcal{M}) \overset{\downarrow}{=} Z^{tw}_{B}(\overline{\mathcal{M}} \boxtimes_{A} \mathcal{M})$

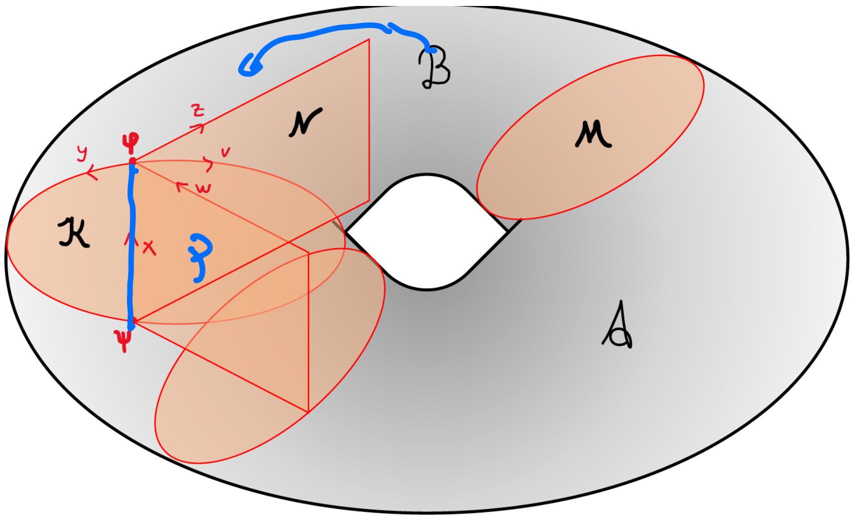
$\mathcal{A} = \mathcal{B}$ and $\mathcal{M} = \mathcal{A}$, $\mathcal{A}$ pivotal $\quad Z_{\mathcal{A}}(\mathcal{A})$

Drinfeld center is braided : it is profitable to consider traces in a 3d setting !

Chapter 3

State-sum models with defects

State-sum models with defects



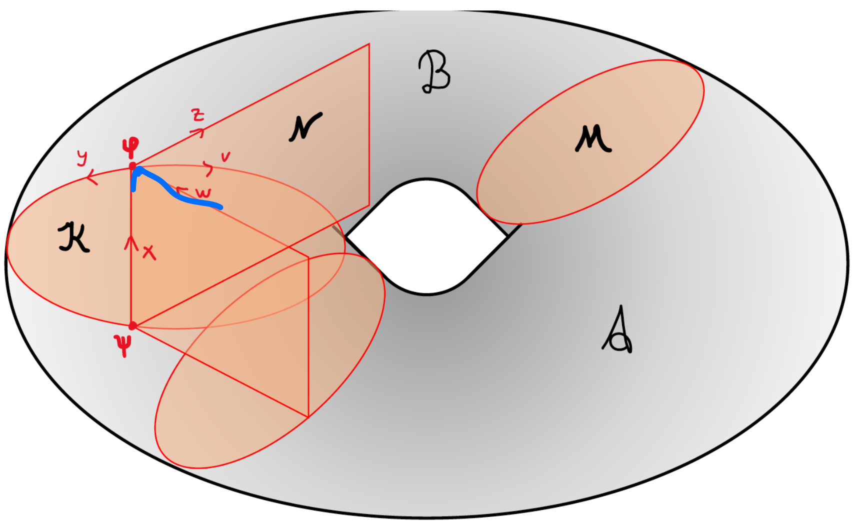
Closed oriented 3-manifold with skeleton

$\mathcal{A}, \mathcal{B}$ spherical fusion categories
$\mathcal{M}, \mathcal{N}$ bimodule categories
Edges: balanced Deligne products

$\mathcal{K} \boxtimes_{\mathcal{B}} \mathcal{N} \boxtimes_{\mathcal{A}} \mathcal{P} \boxtimes_{\mathcal{A}'}$

$\mathcal{A} \boxtimes_{\mathcal{A}} \mathcal{A} \boxtimes_{\mathcal{A}} = Z(\mathcal{A})$

State-sum models with defects



Closed oriented 3-manifold with skeleton

$\mathcal{A}, \mathcal{B}$ spherical fusion categories
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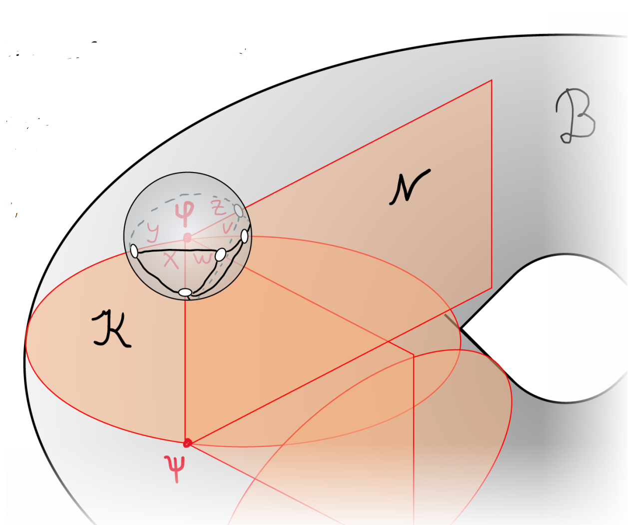
Edges: balanced Deligne products

$$\mathcal{Z}(A) = \mathrm{End}_{A|A}(A)$$

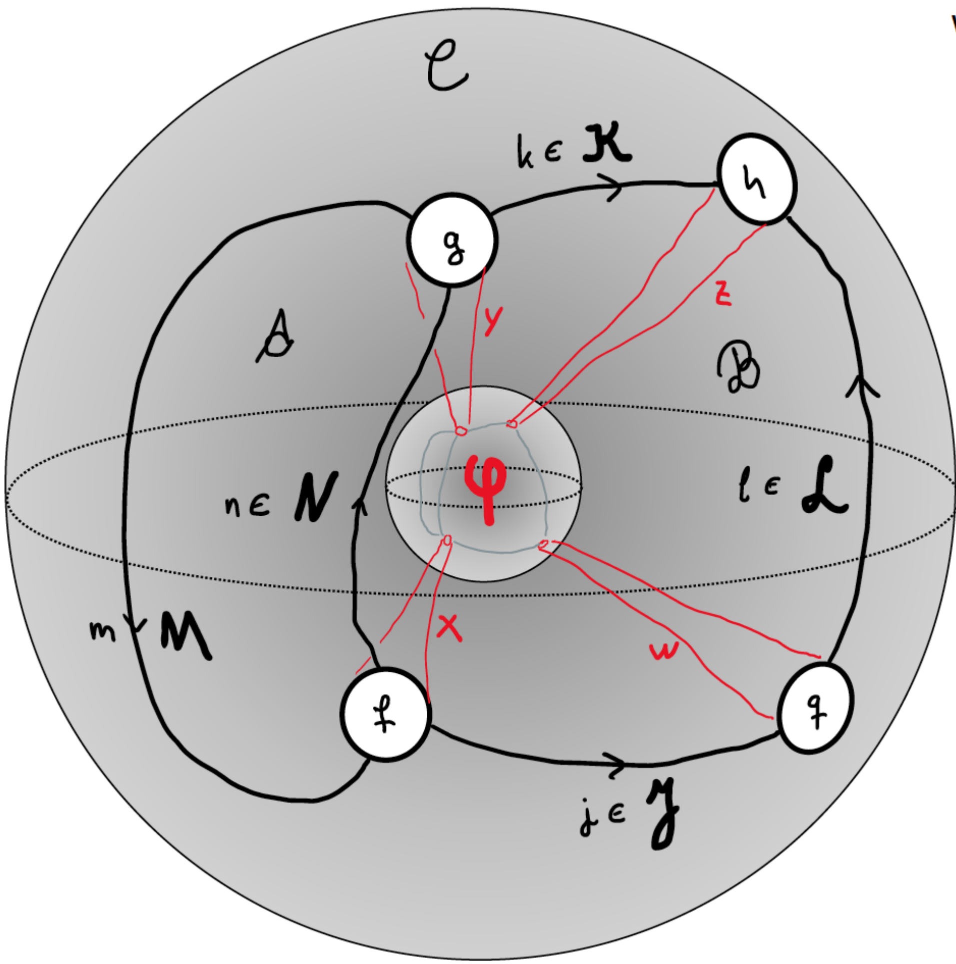


Part of the TV construction is an evaluation at vertices

Chapter 4

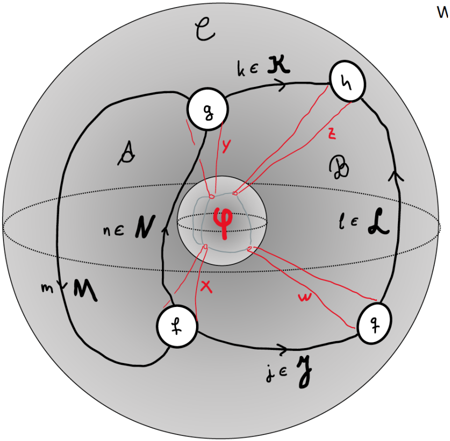
Extruded graphs

Extruded graphs



W

Extruded graphs



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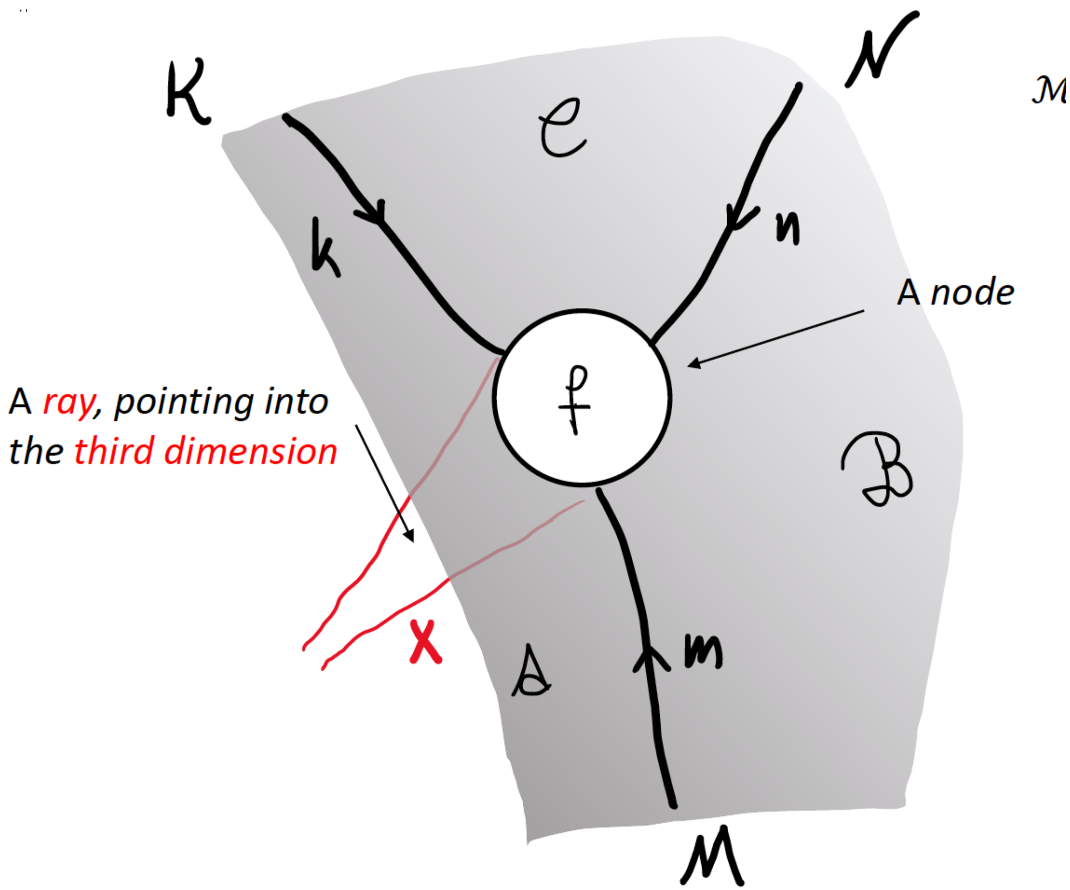
Artin, Theory of braids (1947):
Projection [..] which is an excellent tool for intuitive investigations is a very clumsy one for proofs. This has lead me to abandon projections altogether.

Node labels



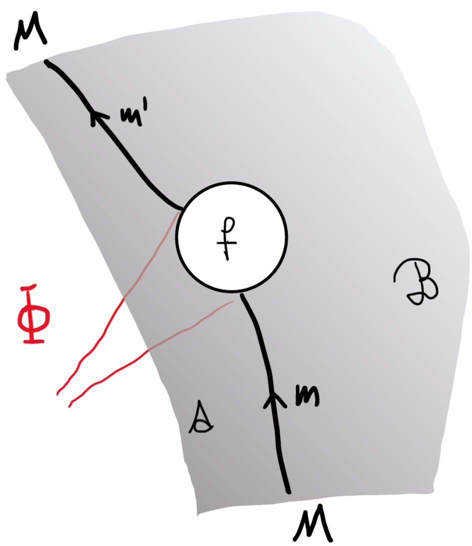
$$x \in \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{K} \boxtimes_{\mathcal{A}}$$

$$U(x) \in \mathcal{M} \boxtimes \mathcal{N} \boxtimes \mathcal{K}$$

$$f : m \boxtimes n \boxtimes n \boxtimes k \to U(x)$$

Sanity check: silent rays

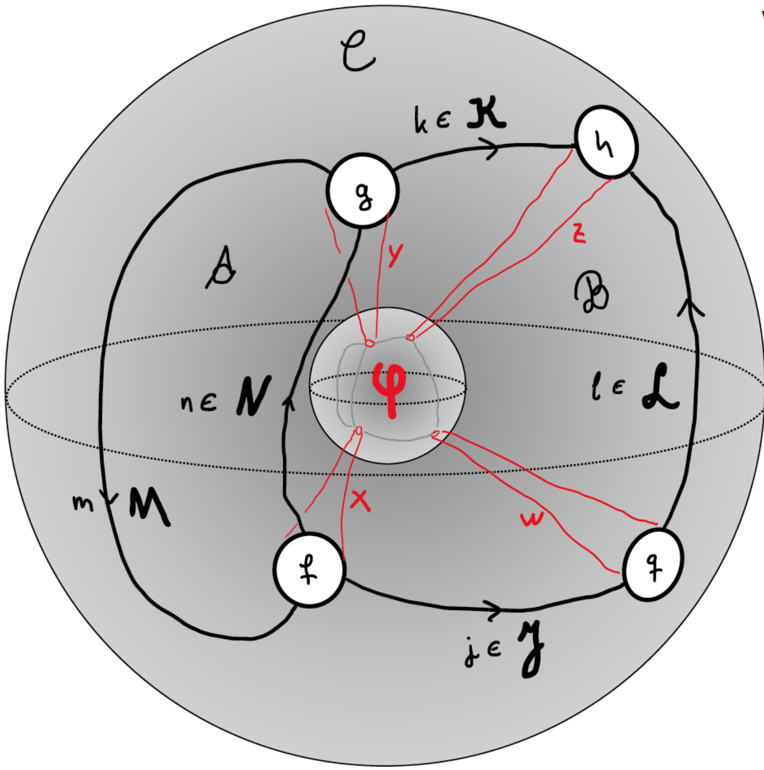
$\mathcal{M} \boxtimes \overline{\mathcal{M}}$



$$U(\Phi) = \bigoplus_\mu \mu \boxtimes \overline{\mu}$$

$$\begin{aligned}
& \mathrm{Hom}_{\mathcal{M} \boxtimes \overline{\mathcal{M}}}(m \boxtimes \overline{m'}, \textstyle\bigoplus \mu \boxtimes \overline{\mu}) \\
= \; & \textstyle\bigoplus_\mu \mathrm{Hom}_{\mathcal{M}}(m, \mu) \otimes \mathrm{Hom}_{\mathcal{M}}(\mu, m') \\
= \; & \mathrm{Hom}_{\mathcal{M}}(m, m')
\end{aligned}$$

Core label

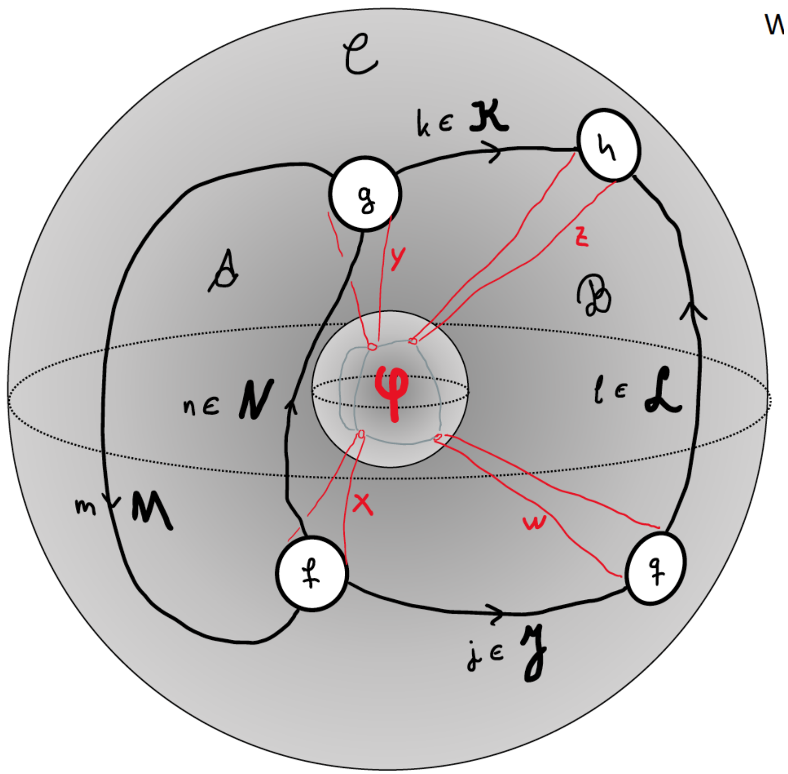


- T^p preblock space, depends on skeleton
- T block spaces, independent
- $T \subset T^p$
- Node labels combine to element α in T^p
- Core label $\varphi \in T^*$

$\varphi(\alpha)$

Use pivotal structure to exhibit T as a retract, $T^p \to T$

Core label



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- T block spaces, independent
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- Node labels combine to element A in T^p
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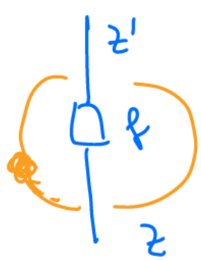
Use pivotal structure to exhibit T as a retract, $T^p \underset{\pi}{\rightarrow} T$

$\varphi(\pi A)$

Prototype: $U : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$

$$\mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(z, z') \subset \mathrm{Hom}_{\mathcal{C}}(Uz, Uz')$$

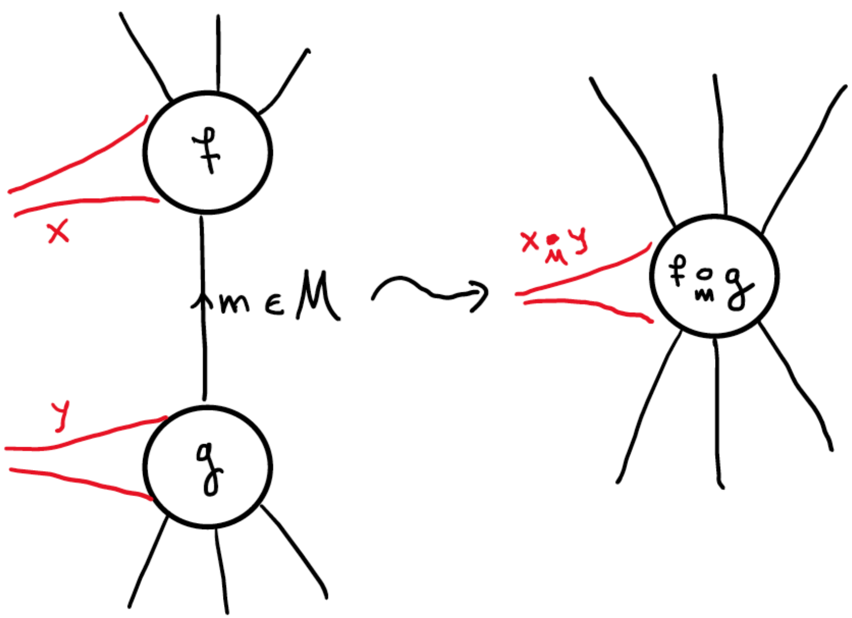


$\longleftarrow\!\!|$ f

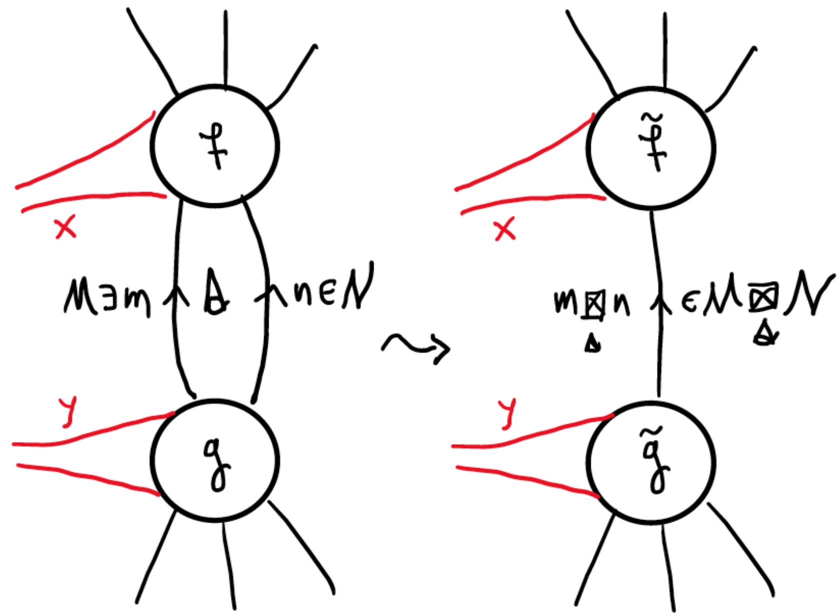
halfbraiding
+ pivotal structure

Local moves

The contraction move



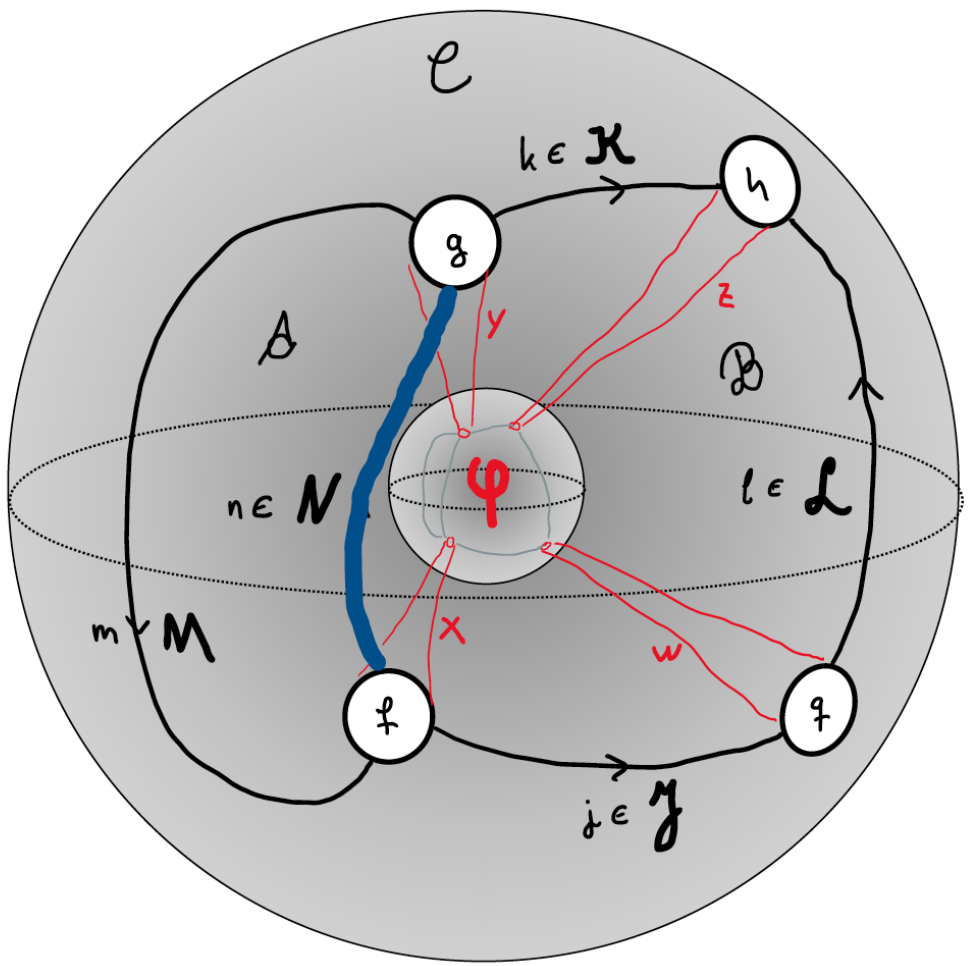
The edge fusion move



Theorem: Performing these changes locally in an extruded graph leaves the evaluation invariant.

Example for an evaluation

Simplifying extruded graphs using moves



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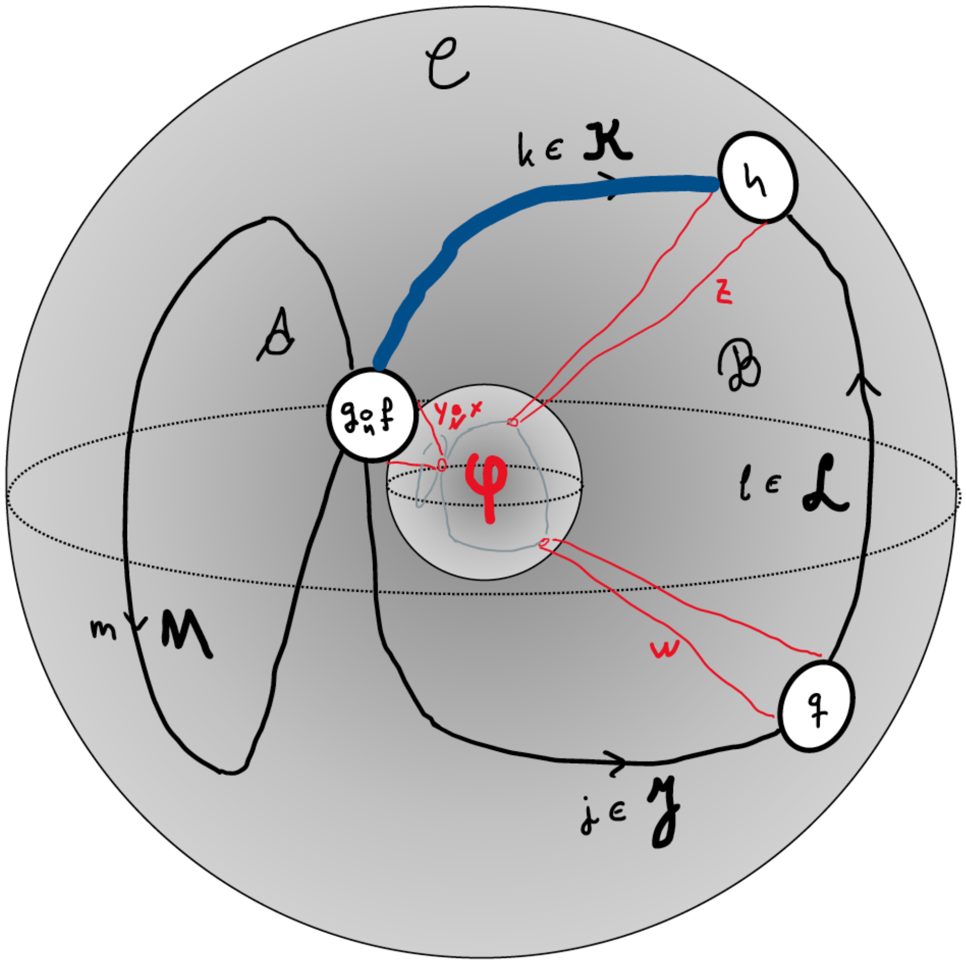

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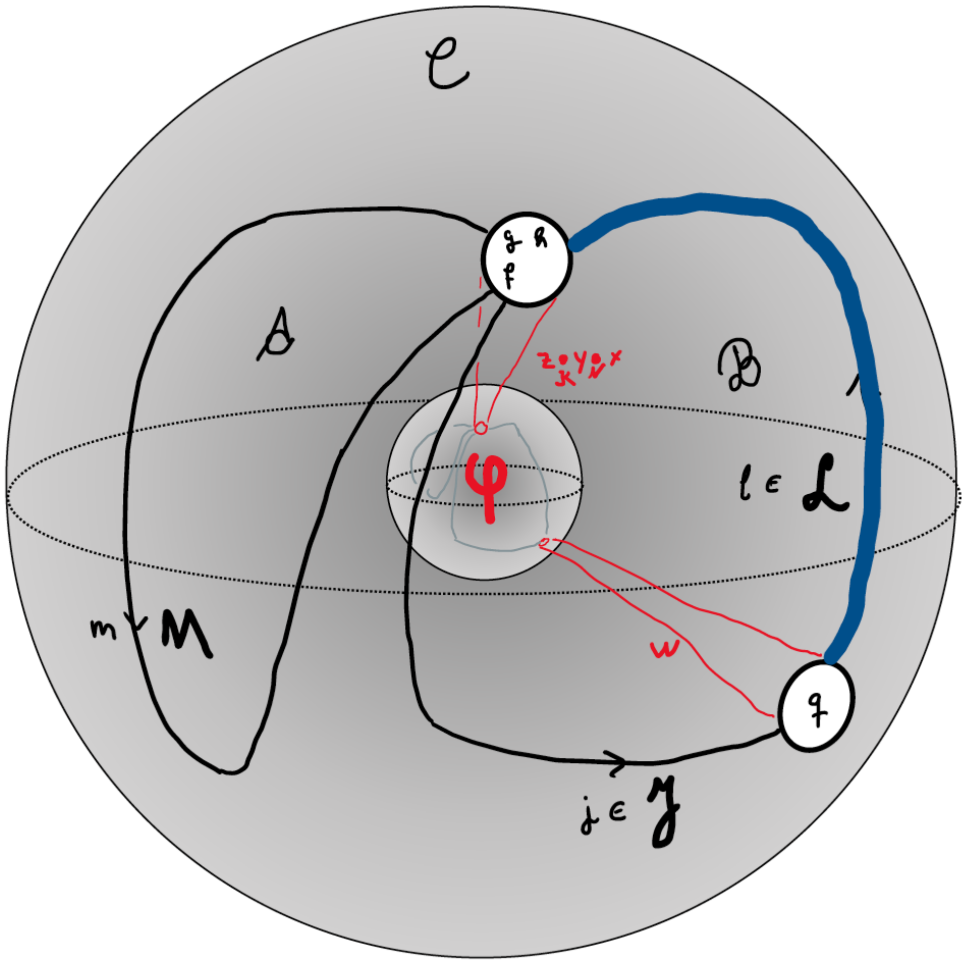

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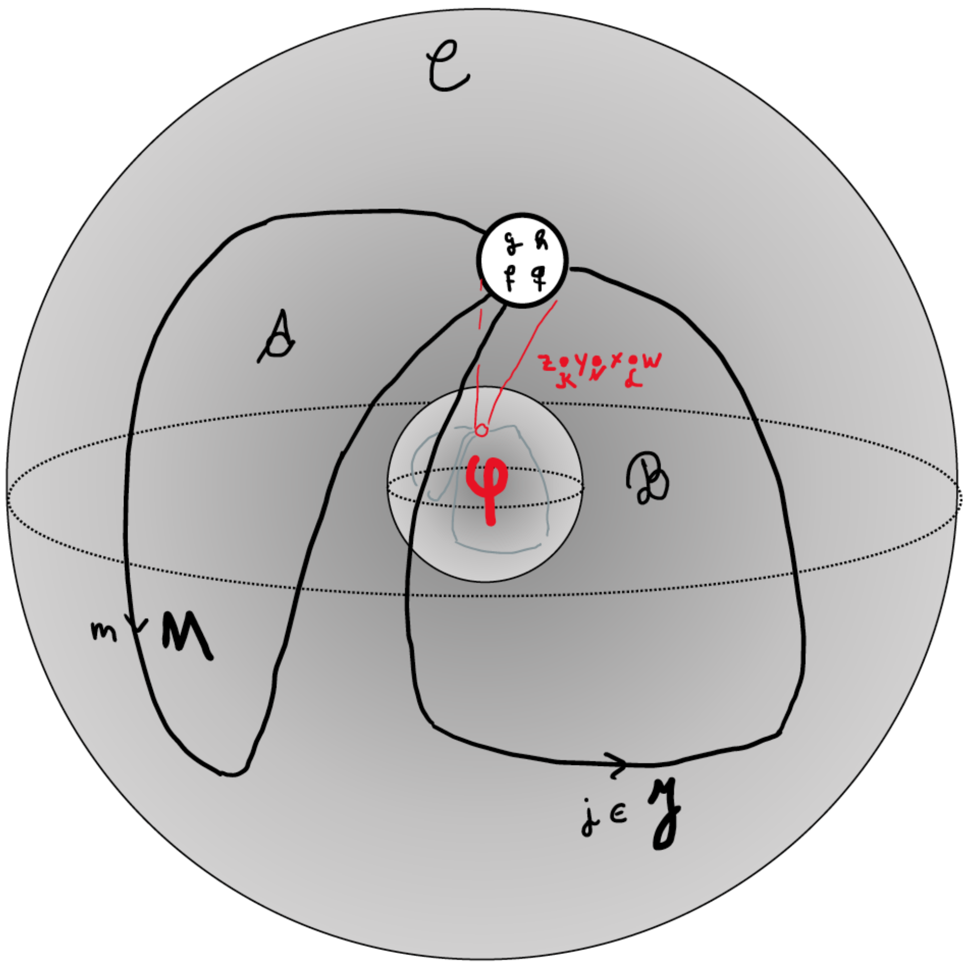

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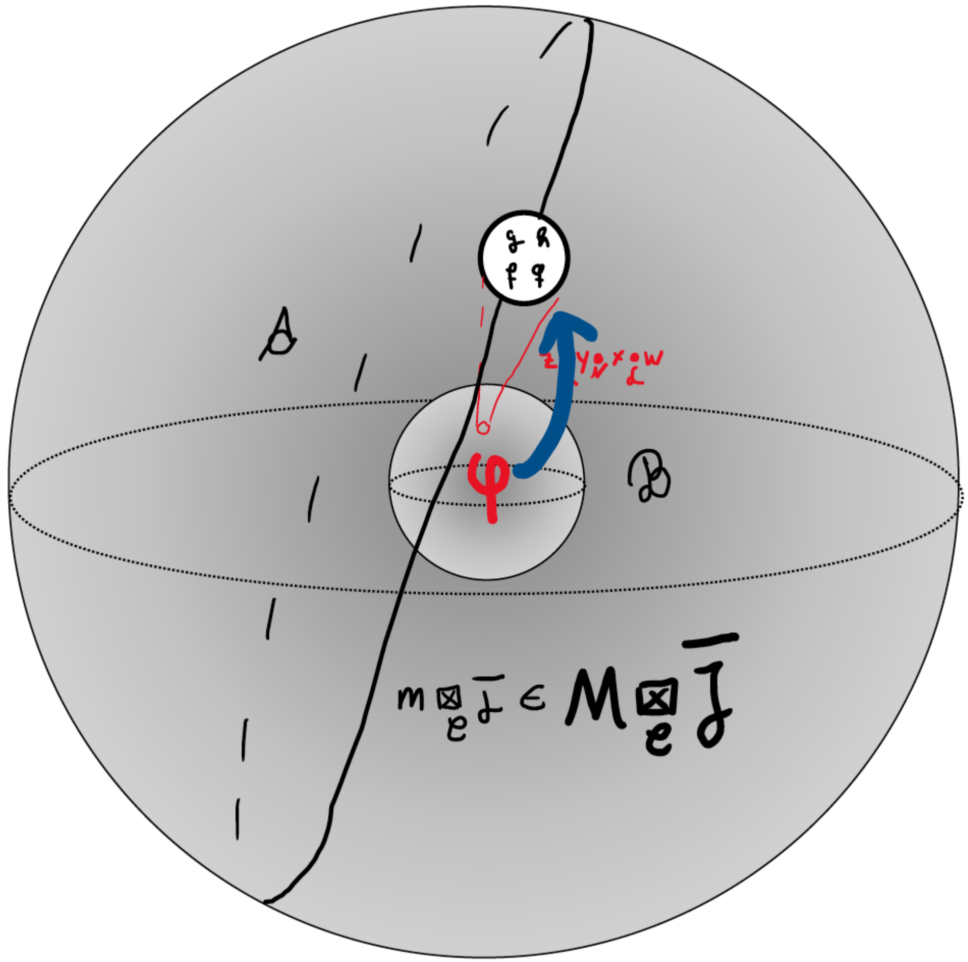
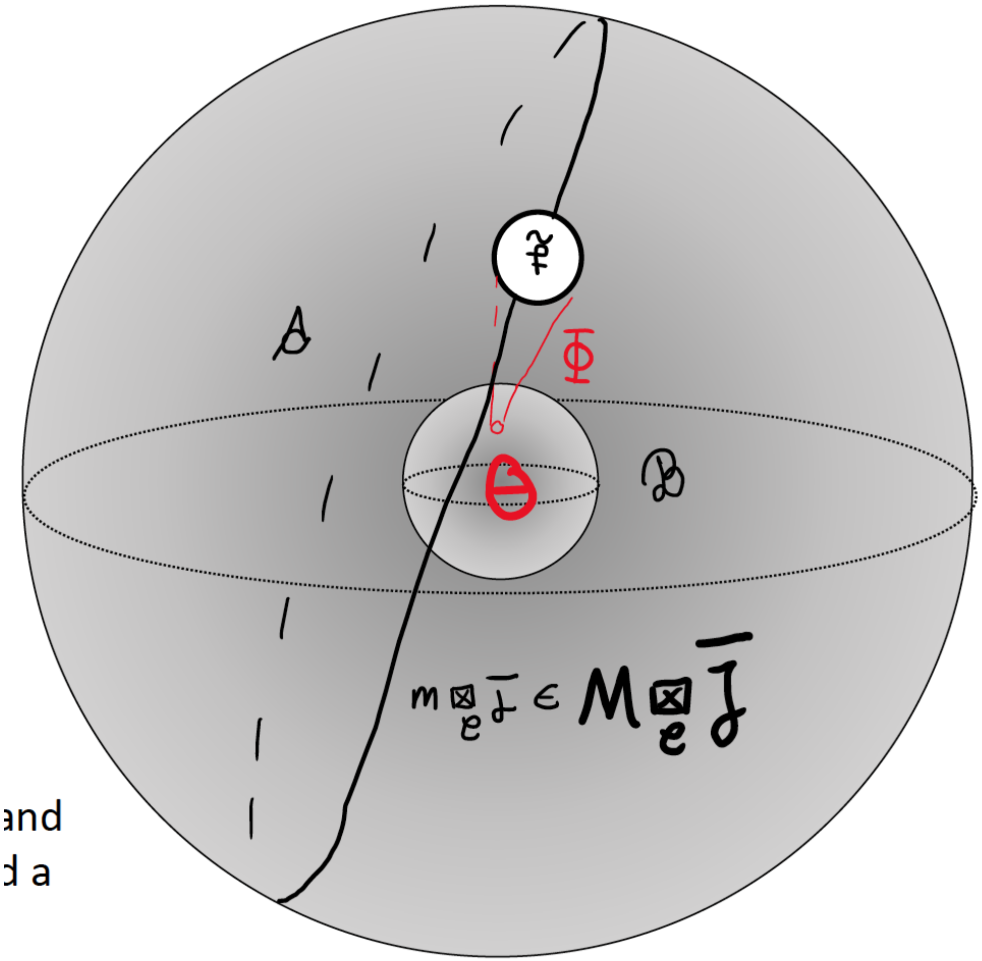


Lift the core label to the coat, leave a silent ray.

Example for an evaluation



$\xrightarrow{\text{evaluation}}$ $\mathrm{Tr}(\tilde{f})$

and

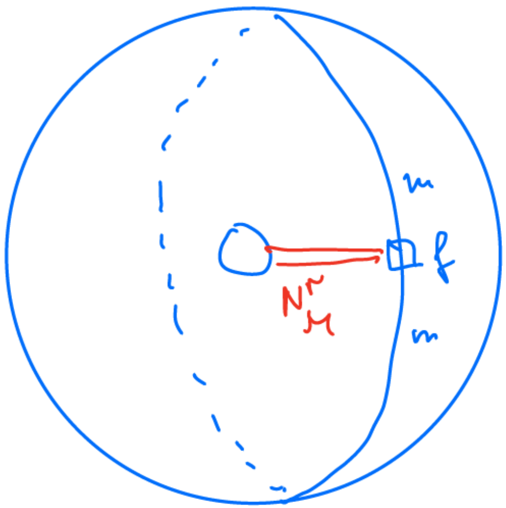
d a

$N^r \cong id$

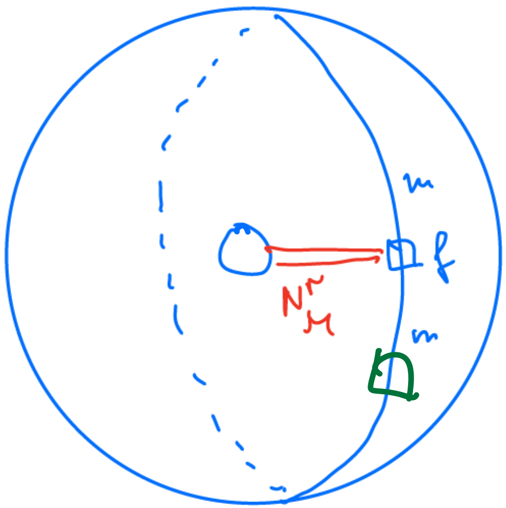
ss. pivotal

$N^r(c) = D_\bullet \, a^{\vee\vee}$

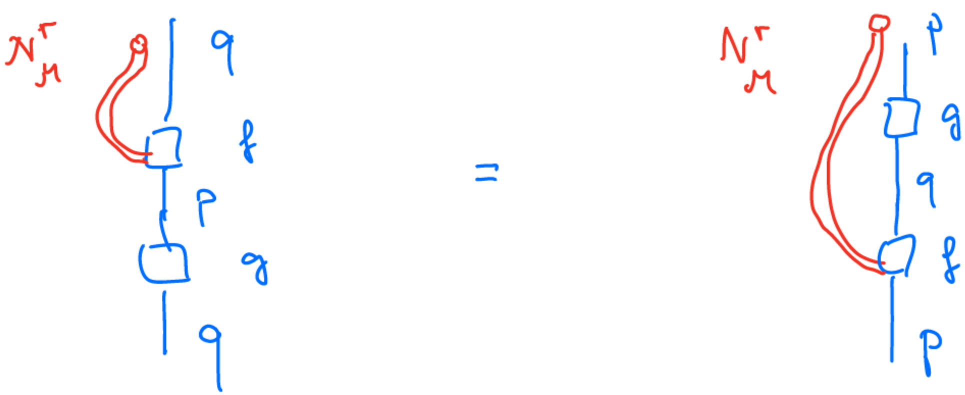
Back to Nakayama functors



Back to Nakayama functors



Cyclicity



Conclusions

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Summary

- Traces are not for endomorphisms and secretely higher structure.
- There is a natural "mildly three-dimensional" graphical calculus for Turaev Viro theories with defects.

Outlook

- The calculus can be derived in the case of Dijkgraaf-Witten theories from field theory.
- There are other interesting higher-dimensional graphical calculi.