Classification results for Hermitian non-Kähler gravitational instantons

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Einstein 4-manifolds

We are in general interested in Einstein 4-manifolds (M, h) for lots of different reasons: Geometry, PDE, Physics...

Various settings and questions:

- Compact case, complete non-compact case.
- Classification in more special settings: hyperkähler, Kähler, self-dual...
- Non-special examples...
- Moduli, degeneration...

Gravitational instantons

In this talk we focus on ***Gravitational instantons***, which are defined to be *complete non-compact Ricci-flat 4-manifolds* (M, h) *with finite* $\int_M |Rm|_h^2 < \infty$.

- Only one end by Cheeger-Gromoll.
- Volume growth $\mathrm{Vol}(B_r) \leq Cr^4$ by volume comparison.
- Curvature decay $|\nabla^k Rm_h|_h \leq Cr^{-2-k}$ by Cheeger-Tian.
- **Assume simply-connected** to avoid other issues.

Goal of today's talk:

- Give a classification of some special class of gravitational instantons.

Curvature decomposition in 4d

The curvature tensor Rm in 4d gives an automorphism of $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$, which decomposes to

$$\begin{pmatrix} \frac{s}{12} + W^+ & Ric^0 \\ Ric^0 & \frac{s}{12} + W^- \end{pmatrix}$$

- $W^+ : \Lambda^+ \to \Lambda^+$ and $W^- : \Lambda^- \to \Lambda^-$ are the self-dual Weyl curvature and the anti-self-dual Weyl curvature.
- s is the scalar curvature and Ric^0 is the traceless Ricci curvature.
- Curvature of gravitational instantons $Rm = \begin{pmatrix} W^+ & 0 \\ 0 & W^- \end{pmatrix}$.

Rough classification

Consider an **oriented** gravitational instanton (M, h).

Theorem (Derdzinski, 1983)

For an oriented gravitational instanton (M, h), it must be one of the following cases:

- *Type I:* $W^+ \equiv 0$.
- *Type II:* $W^+ : \Lambda^+ \to \Lambda^+$ *does not vanish anywhere, and has* ***repeated eigenvalue****, i.e. one repeated eigenvalue and one non-repeated eigenvalue everywhere.*
- *Type III:* W^+ *does not vanish anywhere, and generically has three distinct eigenvalues.*

- Example: Taub-NUT space is Type I under the hyperkähler orientation, while it is Type II under the opposite orientation.

Rough classification

More precisely for an oriented gravitational instanton (M, h) we have:

- Type I $\Leftrightarrow$ (M, h) anti-self-dual/locally hyperkähler $\Leftrightarrow$ (M, h) globally hyperkähler by simply-connectedness.
- Type II $\Leftrightarrow$ (M, h) is Hermitian non-Kähler, but conformally Kähler.
- Type III $\Leftrightarrow$ (M, h) is never conformally Kähler with the given orientation.

Type II in more details

For (M, h) a Type II gravitational instanton

- Set

$$\lambda = 2\sqrt{6}|W^+|_h, \quad g = \lambda^{2/3} h,$$
$$\omega = \text{eigen-2-form of } W^+ \text{ of the non-repeated eigenvalue.}$$

  Then g is Kähler with ω as the Kähler form, and J is determined by g and ω. Scalar curvature of g is $s_g = \lambda^{1/3}$.

- (M, g, J) is actually Kähler extremal, i.e. the vector field $\nabla_g^{1,0} s_g$ is a Hamiltonian Killing field.

- (M, h, J) is Hermitian non-Kähler.

Compact analogue

Theorem (LeBrun, 2012)

A compact Hermitian non-Kähler Einstein 4-manifold (M, h) *must either be:*

- *the Page metric on* $Bl_p\mathbb{P}^2$.
- *or the Chen-LeBrun-Weber metric on* $Bl_{p,q}\mathbb{P}^2$.

- They are conformal to Kähler extremal metrics as above. They are all **toric**.
- The Chen-LeBrun-Weber metric is discovered by them in 2008.
- In particular, no compact Hermitian non-Kähler Ricci-flat 4d metric.

Type I gravitational instantons

Type I gravitational instantons (with the simply-connected assumption) are just hyperkähler gravitational instantons.

- Examples by Atiyah-Hitchin, Cherkis-Kapustin, Tian-Yau, Biquard-Minerbe, Hein...
- Classification by Kronheimer, Minerbe, Chen-Chen, Sun-Zhang, Chen-Viaclovsky, Chen-Viaclovsky-Zhang...

Notably, they all must be asymptotic to certain **model spaces**, the ALE, ALF, ALG, ALH, ALG*, ALH* model spaces, at infinity by Sun-Zhang (2021).

Asymptotic to model spaces

A gravitational instanton (roughly) is said to be

- ALE if asymptotic to $\mathbb{R}^4/\Gamma$.
- ALF if asymptotic to the models $Vg_{\mathbb{R}^3} + V^{-1}\eta^2$ where $V = (k+1)/2r + c$ and $d\eta = \star dV$ (or its $\mathbb{Z}_2$ quotient).
- ALG if asymptotic to a T^2-fibration over a 2d base.
- ALH if asymptotic to a T^3-fibration over $(0, \infty)$.
- There are two exceptional models ALG* and ALH*.

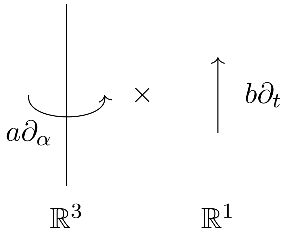
Hyperkähler gravitational instantons must be asymptotic to one of the above models by Sun-Zhang (2021).

Type II gravitational instantons

One more model space for Type II that cannot happen for hyperkähler gravitational instantons:

Definition (AF model spaces)

A gravitational instanton is $\mathrm{AF}_{a,b}$ if it is asymptotic to $(\mathbb{R}^3 \times \mathbb{R}^1)/\mathbb{Z}$, where $\mathbb{Z}$ action generated by $(\rho, \alpha, \beta, t) \mapsto (\rho, \alpha + a, \beta, t + b)$. Here $0 \leq \alpha < 2\pi$ and $0 \leq \beta \leq \pi$ are the spherical coordinates. ρ the distance function on $\mathbb{R}^3$.



Type II gravitational instantons

The $\mathrm{AF}_{a,b}$ model $(\mathbb{R}^3 \times \mathbb{R}^1)/\mathbb{Z}$ with $(\rho, \alpha, \beta, t) \sim (\rho, \alpha + a, \beta, t + b)$:

- The model space $(\mathbb{R}^3 \times \mathbb{R}^1)/\mathbb{Z}$ goes back to Gromov and referred to as Gromov's skew rotation.
- It has $\mathbb{R}^3/\mathbb{Z}$ with $(\rho, \alpha, \beta) \sim (\rho, \alpha + a, \beta)$ as tangent cone when $a/2\pi$ rational.
- It has the right half plane as tangent cone when $a/2\pi$ is irrational, parametrized by (ρ, β).
- The geometry is not well-understood in the irrational case. We don't have an explicit estimate for injective radius at infinity. Minerbe showed $inj(t) \leq C\sqrt{t}$.

Type II gravitational instantons: examples

Type II ⇔ Hermitian non-Kähler, but conformally Kähler extremal. The Kähler metric $g = \lambda^{2/3}h$ with $\lambda = 2\sqrt{6}|W^+|_h$.

Example (Reversed Taub-NUT, ALF)

- The usual Taub-NUT metric can be written as $h = \frac{4\rho+1}{\rho}d\rho^2 + \rho(\rho+1)(\sigma_1^2 + \sigma_2^2) + \frac{\rho}{\rho+1}\sigma_3^2$ on $\mathbb{R}_+ \times S^3$.
- Take the complex structure $J : \sigma_2 \to \sigma_1, \sigma_3 \to \frac{\rho+1}{2\rho}d\rho$. This J is **not** one of the hyperkähler complex structures and the orientation is opposite to the hyperkähler one.

$$|W^+|_h = \frac{C}{\rho^2(\rho+1)}$$

$$\lambda^{1/3} = C\left(\frac{1}{\rho^2(\rho+1)}\right)^{1/3} \sim \frac{C}{\rho}.$$

Conformal change $g = \lambda^{2/3}h$ gives a Kähler extremal on $\mathbb{C}^2$ with Poincáre behavior at infinity.

Type II gravitational instantons: examples

Example (Reversed Eguchi-Hanson, ALE)

- The Eguchi-Hanson metric can be written as $h = \frac{1}{1-(\frac{a}{\rho})^4} d\rho^2 + \rho^2 \left(\sigma_1^2 + \sigma_2^2 + (1 - (\frac{a}{\rho})^4)\sigma_3^2 \right)$ on $(a, \infty) \times S^3$.
- Take the complex structure

$$J : \sqrt{\frac{1}{1-(a/\rho)^4}} d\rho \to \rho \sqrt{1-(a/\rho)^4} \sigma_3, \ \sigma_2 \to \sigma_1.$$

This is **not** one of the hyperkähler complex structures and the orientation is opposite to the hyperkähler one.

Conformal change $g = \lambda^{2/3} h$ gives an incomplete Kähler extremal on $\mathcal{O}(2)$, extending to a Kähler extremal orbifold on $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2))$ with $C^2 = -2$ shrunk.

Type II gravitational instantons: examples

Example (Kerr, AF)

The Kerr family on $\mathbb{C}^1 \times \mathbb{P}^1$ is a 1-parameter family of Ricci-flat metrics that are AF, containing the Schwarzschild metric.

Example (Chen-Teo, AF, 2011)

The Chen-Teo family on $Bl_p(\mathbb{C}^1 \times \mathbb{P}^1)$ is a 1-parameter family of Ricci-flat metrics that are AF. Constructed by Chen-Teo using inverse scattering method. Observed to be Hermitian non-Kähler (i.e. Type II) by Aksteiner-Andersson (2021).

There is also the Taub-bolt metric that is ALF. Type II in both orientations.

The corresponding Kähler extremal metrics are all of Poincáre type.

Type II gravitational instantons: previous results

- Biquard-Gauduchon (2021) classified all toric Hermitian non-Kähler ALF/AF gravitational instantons.
- Aksteiner-Andersson further conjectured the reversed Taub-NUT, Kerr, Chen-Teo, Taub-bolt metrics are all the Type II ALF/AF gravitational instantons, in comparison to the compact case.
  Compact Hermitian non-Kähler 4d Einstein metrics are **all toric** due to LeBrun (1995).

Type II ALE gravitational instantons

Theorem (Compactifications of Type II ALE, L. 2023a)

- *For a Type II ALE gravitational instanton (M, h), a suitable end is biholomorphic to B^*/Γ, where $B^* \subset \mathbb{C}^2$ the standard punctured ball and $\Gamma \subset U(2)$.*
- *The Kähler metric (M, g) can be compactified to a Kähler extremal orbifold $(\widehat{M}, \widehat{g})$. The underlying complex orbifold $\widehat{M}$ is log del Pezzo.*

Theorem (Partial classification, L. 2023a)

Any Type II ALE gravitational instanton with $\Gamma \subset SU(2)$ must be the reversed Eguchi-Hanson.

The technique actually extends to the $\Gamma \subset U(2)$ case but needs more calculations.

Type II ALF/AF gravitational instantons

Theorem (Toricness of Type II ALF/AF, L. 2023b)

Any Type II ALF/AF gravitational instanton must be toric.

As a corollary we conclude the entire classification of Type II ALF/AF gravitational instantons and confirms the conjecture by Aksteiner-Andersson

Corollary (Biquard-Gauduchon 2021, L. 2023b)

Any Type II ALF/AF gravitational instanton must be one of the reversed Taub-NUT, Taub-bolt, Kerr, Chen-Teo metrics.

Type II ALF/AF gravitational instantons

Theorem (Compactifications of Type II ALF/AF, L. 2023b)

All Type II ALF gravitational instantons and $AF_{a,b}$ gravitational instantons with rational $a/2\pi$ can be naturally compactified to a toric log del Pezzo surfaces by adding a $\mathbb{P}^1$.

- A Calabi-type theorem gives holomorphically toric $\Rightarrow$ holomorphically isometrically toric.
- The extremal Kähler metric asymptotically locally splits as the product of a sphere with a cusp (first observed by Biquard-Gauduchon in the toric case). The compactification is done by compactifying each cusp direction.
- LeBrun (1995) proved compact Hermitian non-Kähler 4d Einstein metric only resides on toric del Pezzo surfaces.

Type II gravitational instantons with more collapsing

Theorem (Non-existence with severer collapsing, L. 2023b)

There is no Type II gravitational instantons, with curvature decay $o(\rho^{-2})$, which only have the following two kinds of tangent cones:

- *1-dimensional tangent cones.*
- *Smooth 2-dimensional tangent cones.*

- This non-existence is **sharp**, as Type II irrational AF gravitational instantons have the right-half-plane as tangent cone, which is not smooth.
- This is in *sharp contrast* to the hyperkähler case. Hyperkähler gravitational instantons all have smooth unique tangent cone and all have curvature decay $o(\rho^{-2})$ except the ALH* case. Our result demonstrates in most of the cases there is no Type II gravitational instanton.

Gravitational instantons: recap

<table>
<tr><th>Model</th><th>Type I
hyperkähler</th><th>Type II
Hermitian non-Kähler</th><th>Type III</th></tr>
<tr><td>ALE</td><td>Kronheimer 89</td><td>No if $\Gamma \subset SU(2)$
L. 23a</td><td>Nakajima
conjecture</td></tr>
<tr><td>ALF</td><td>Minerbe 09
Chen-Chen 15</td><td>Biquard-Gauduchon 21
L. 23b</td><td rowspan="7">?</td></tr>
<tr><td>AF</td><td>Not applicable</td><td>Biquard-Gauduchon 21
L. 23b</td></tr>
<tr><td>ALG</td><td>Chen-Chen 15</td><td rowspan="3">Does not exist
L. 23b</td></tr>
<tr><td>ALH</td><td>Chen-Chen 15</td></tr>
<tr><td>ALG*</td><td>Chen-Viaclovsky
-Zhang 21</td></tr>
<tr><td>ALH*</td><td>Collins-Jacob
-Lin 21</td><td>?</td></tr>
</table>

Gravitational instantons: further questions

Questions:

- Complete the $\Gamma \subset U(2)$ case for Type II ALE.
- Type II ALH* gravitational instantons?
- Classification for asymptotic models of Type II as Sun-Zhang? Is there other exceptional model for Type II?
- Say anything nontrivial about the Type III case? This can be compared to the compact situation: is there a compact 4d Ricci-flat metric that is not of special holonomy/Kähler.

Gravitational instantons: recent results

Other recent results:

- Aksteiner-Andersson et al. 2023 studied ALF/AF gravitational instantons with S^1 symmetry. They showed when the topology is Kerr/Taub-bolt, then it must be Kerr/Taub-bolt. Similar results for Chen-Teo.
- Biquard-Gauduchon-LeBrun 2023 studied the deformation of Type II ALF/AF gravitational instantons.
- Biquard-Ozuch 2023 studied the stability of Type II ALF/AF gravitational instantons.

Sketch of non-existence of Type II with more collapsing

(M, h) Type II, $\lambda = 2\sqrt{6}|W^+|_h$, $g = \lambda^{2/3}h$ the Kähler extremal metric. Further assume $Rm_h = o(\rho^{-2})$. Take a tangent cone $(M, h_i, p) \to (M_\infty, d_\infty, p_\infty)$.

- Pick small ball $B_i \to B_\infty$ that is away from the cone vertex p

$$\begin{array}{ccc}
(\widetilde{B_i}, \widetilde{h_i}, G_i) & \xrightarrow{\textit{equivariant } GH} & (\widetilde{B_\infty}, \widetilde{h_\infty}, G_\infty) \\
\pi_i \downarrow & & \downarrow \pi_\infty \\
(B_i, h_i) & \xrightarrow[GH]{} & (B_\infty, h_\infty).
\end{array}$$

  The convergence upstairs is smooth and non-collapsing. The limit $\widetilde{B_\infty}$ is flat.

- Key: capture the information of the Kähler metric g.

Type II gravitational instantons with more collapsing

Step 1. Rescale the conformal factor to λ_i. Each $g_i = \lambda_i^{2/3} h_i$ is Kähler extremal.

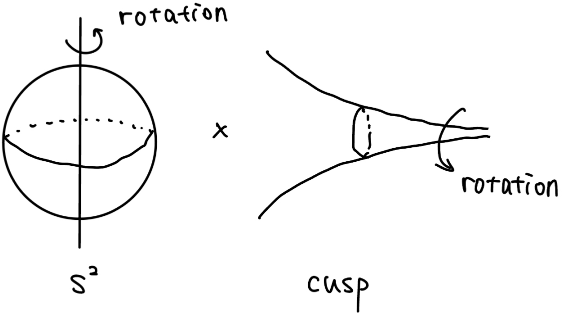
Step 2. Harnack inequality for the rescaled conformal factor $\lambda_i^{2/3}$. Pulled back to $\widetilde{B_i}$, $\lambda_i^{2/3}$ converges smoothly to $\lambda_\infty^{2/3}$ by elliptic theory.

$$\begin{array}{ccc} (\widetilde{B_i}, \widetilde{h_i}, \lambda_i, g_i, G_i) & \xrightarrow{\text{equivariant } GH} & (\widetilde{B_\infty}, \widetilde{h_\infty}, \lambda_\infty, g_\infty, G_\infty) \\ \pi_i \downarrow & & \downarrow \pi_\infty \\ (B_i, h_i) & \xrightarrow[GH]{} & (B_\infty, h_\infty). \end{array}$$

Type II gravitational instantons with more collapsing

Step 3. The flat metric $(\widetilde{B_\infty}, \widetilde{h_\infty})$ is conformal to $g_\infty = \lambda_\infty^{2/3}\widetilde{h_\infty}$ that is Kähler. This forces g_∞ to split as $S^2 \times$ cusp.

Step 4. The symmetry G_∞ in particular cannot be 3-dimensional, which concludes the non-existence of 1-dimensional tangent cone.



Thank you for your attention!