Local $U(2,2)$ Symmetry in Relativistic Quantum Mechanics[*]

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Abstract

Local gauge freedom in relativistic quantum mechanics is derived from a measurement principle for space and time. For the Dirac equation, one obtains local $U(2,2)$ gauge transformations acting on the spinor index of the wave functions. This local $U(2,2)$ symmetry allows a unified description of electrodynamics and general relativity as a classical gauge theory.

1 Connection between Local Gauge Freedom and Position Measurements

In [3], it was suggested to link the physical gauge principle with quantum mechanical measurements of the position variable. In the present paper, we will extend this concept to relativistic quantum mechanics and apply it to the Dirac equation. For Dirac spinors, we obtain local $U(2,2)$ gauge freedom. Our main result is that this $U(2,2)$ symmetry allows a natural description of both electrodynamics and general relativity as a classical gauge theory. This is shown by deriving a $U(2,2)$ spin connection from the Dirac operator and analyzing the geometry of this connection. Although we develop the subject from a particular point of view, this paper can be used as an introduction to the Dirac theory in curved space-time.

In contrast to [3], where the point of interest is the measure theoretic derivation of local gauge transformations, we will here concentrate on the differential geometry of the Dirac operator. In order to keep measure theory out of this paper, we will use a bra/ket notation in position space. This allows us to explain the basic ideas and results of [3] in a simple, non-technical way which will be sufficient for the purpose of this paper. Nevertheless one should keep in mind that the bra/ket symbols and the δ-normalizations are only a formal notation; the mathematical justification for this formalism is given in [3].

We begin with the example of a scalar particle in nonrelativistic quantum mechanics. The particle is described by a wave function $\Psi(\vec{x})$, which is a vector of the Hilbert space $H = L^2(\mathrm{I\!R}^3)$. The physical observables correspond to self-adjoint operators on H. The position operators $\vec{X}$, for example, are given as multiplication operators with the coordinate functions, $X^i : \Psi(\vec{x}) \to x^i \, \Psi(\vec{x})$, $i = 1, \ldots, 3$. Our definition of the Hilbert space as a space of functions in the position variable was only a matter of convenience; e.g. we could just as well have introduced H as functions in momentum space. Therefore it seems reasonable to forget about the fact that H is a function space and consider it merely as

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an abstract Hilbert space. After this generalization, which is often implicitly assumed in quantum mechanics, we must construct the representation of H as wave functions. For this purpose, we choose an "eigenvector basis" $|\vec{x}>$ of the position operators,

$$X^i \, |\vec{x}> \; = \; x^i \, |\vec{x}> \quad , \qquad <\vec{x} \, | \, \vec{y}> \; = \; \delta^3(\vec{x} - \vec{y}) \qquad \qquad , \qquad (1)$$

and define the wave function of a vector $\Psi \in H$ by $\Psi(\vec{x}) = <\vec{x} \, | \, \Psi>$.

This "position representation" of the Hilbert space is quite elementary and is currently used in physics. We point out that it is not unique, because the "eigenvectors" $|\vec{x}>$ are only determined up to a phase. Namely, we can transform $|\vec{x}>$ according to

$$|\vec{x}> \; \rightarrow \; e^{ie\Lambda(\vec{x})} \, |\vec{x}> \qquad\qquad (2)$$

with a real function $\Lambda(\vec{x})$. This corresponds to a local phase transformation

$$\Psi(\vec{x}) \; \rightarrow \; e^{-ie\Lambda(\vec{x})} \, \Psi(\vec{x}) \qquad\qquad (3)$$

of the wave functions. The arbitrariness of the local phase of the wave functions can also be understood directly from the fact that the wave function itself is not observable, only its absolute square $|\Psi|^2$ has a physical interpretation as probability density. It is important for the following that the local phase transformations (2),(3) can be interpreted as $U(1)$ gauge transformations. To see this, we consider the canonical momentum operator $\vec{\pi} = -i\vec{\nabla} - e\vec{A}$ with magnetic vector potential $\vec{A}$. Under the transformation (2), the canonical momentum behaves like

$$\vec{\pi} \; \rightarrow \; e^{-ie\Lambda(\vec{x})} \, \vec{\pi} \, e^{ie\Lambda(\vec{x})} \; = \; -i\vec{\nabla} - e(\vec{A} - \vec{\nabla}\Lambda) \qquad\qquad ,$$

which corresponds to the gauge transformation $\vec{A} \rightarrow \vec{A} - \vec{\nabla}\Lambda$ of the vector potential.

In this way, we have explained the local $U(1)$ gauge symmetry of the magnetic field from the fact that space is a quantum mechanical observable. Unfortunately, this natural explanation of local $U(1)$ gauge freedom only works for scalar particles. In the general case with spin, the wave functions have several components, $\Psi \in H = L^2(\mathrm{I\!R}^3)^m$ (i.e. $m = 2s + 1$ for particles with spin s). We again consider H as an abstract Hilbert space. In order to construct the representation of a vector $\Psi \in H$ as a function, we choose an "orthonormal basis" $|\vec{x}\alpha>$, $\vec{x} \in \mathrm{I\!R}^3$, $\alpha = 1, \ldots, m$ of the position operators,

$$X^i \, |\vec{x}\alpha> \; = \; x^i \, |\vec{x}\alpha> \quad , \qquad <\vec{x}\alpha \, | \, \vec{y}\beta> \; = \; \delta_{\alpha\beta} \, \delta^3(\vec{x} - \vec{y}) \qquad\qquad , \qquad (4)$$

and define the wave function by $\Psi^\alpha(\vec{x}) = <\vec{x}\alpha \, | \, \Psi>$. The basis $|\vec{x}\alpha>$ is only unique up to local unitary transformations,

$$|\vec{x}\alpha> \; \rightarrow \; \sum_{\beta=1}^{m} (U^{-1})^\alpha_\beta(\vec{x}) \, |\vec{x}\beta> \qquad \text{with} \qquad U(\vec{x}) \in U(m) \qquad , \qquad (5)$$

as is verified by substituting into (4). Thus the wave functions can be transformed according to

$$\Psi(\vec{x}) \; \rightarrow \; U(\vec{x}) \, \Psi(\vec{x}) \qquad\qquad . \qquad (6)$$

It would be nice if we could again identify the local $U(m)$ transformations (5),(6) with physical gauge transformations. Then the local gauge principle would no longer be an a-priori principle in physics. It would be a consequence of a quantum mechanical "measurement principle", namely the description of space with observables X^i on an

abstract Hilbert space. Notice that the local $U(m)$ transformations (6) are a generalization of the phase transformations (3). Our idea is that the additional degrees of freedom of the larger gauge group in (6) might make it possible to describe additional interactions (like gravitation or the weak and strong forces). Notice that, as a great advantage of our procedure, the local gauge group could no longer be chosen arbitrarily; it would be determined by the configuration of the spinors. Thus we could hope to get into the position to explain the gauge groups observed in physics. At the moment, however, it is not clear if the interpretation of (5),(6) as gauge transformations really makes physical sense. We will in the following simply assume it as a postulate and want to study its further consequences.

In view of our later generalization to curved space (and curved space-time), it is convenient to introduce the spectral measure $dE_{\vec{x}}$ of the position operators: We form the "projectors" $E_{\vec{x}}$ on the "eigenspaces" of $\vec{X}$,

$$E_{\vec{x}} \;=\; \sum_{\alpha=1}^{m} |\vec{x}\alpha><\vec{x}\alpha| \qquad . \tag{7}$$

The $E_{\vec{x}}$ do not depend on the choice of $|\vec{x}\alpha>$; we can characterize them by the operator relations

$$X^i \, E_{\vec{x}} \;=\; x^i \, E_{\vec{x}} \;\;, \qquad E_{\vec{x}} \, E_{\vec{y}} \;=\; \delta^3(\vec{x}-\vec{y}) \, E_{\vec{x}} \qquad . \tag{8}$$

The δ^3-normalization in (8) leads to difficulties as soon as curved coordinate systems are considered. To avoid these problems, it is useful to combine $E_{\vec{x}}$ with the integration measure by introducing the spectral measure $dE_{\vec{x}} := E_{\vec{x}} \, d\vec{x}$. Integrating over the spectral measure yields an operator on the Hilbert space. For example, we have

$$X^i \;=\; \int_{\mathbb{R}^3} x^i \, dE_{\vec{x}} \;\;, \qquad \chi_V \;=\; \int_V dE_x \qquad , \tag{9}$$

where χ_V is the multiplication operator with the characteristic function (i.e. $(\chi_V \Psi)(x)$ is equal to $\Psi(x)$ if $x \in V$ and vanishes otherwise). These relations can be verified directly with the help of (4) and (7). Actually, the spectral measure is completely characterized by (9). In the mathematical paper [3], spectral measures on a manifold are used as the starting point. In this more general approach, the number of components of the wave functions $m = m(x)$ may vary in space. We call it the *spin dimension*.

2 Generalization to Dirac Spinors, $U(2,2)$ Gauge Symmetry

If our explanation of local gauge freedom shall have general significance, it must be possible to extend it to the relativistic setting. We start the analysis with Dirac wave functions in Minkowski space[1]. Since we do not yet want to introduce Dirac matrices and the Dirac operator, we consider the wave functions $\Psi, \Phi, \dots$ just as 4-component functions. We define a scalar product on the spinors,

$$\prec\Psi,\Phi\succ(x) \;=\; \sum_{\alpha=1}^{4} s_\alpha \left(\Psi^\alpha(x)\right)^* \Phi^\alpha(x) \qquad \text{with} \qquad s_1 = s_2 = 1, \;\; s_3 = s_4 = -1 \;\;, \tag{10}$$

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[1]We mention for completeness that *Minkowski space* is a four-dimensional real vector space with a scalar product of signature $(+ - - -)$. A coordinate system where this scalar product coincides with the Minkowski metric η_{ij} is called *reference frame*. We will mainly work in a fixed reference frame and can then identify Minkowski space with $\mathbb{R}^4$.

which is called *spin scalar product.* It is indefinite of signature $(2,2)$. Later, after the introduction of the Dirac matrices in Dirac representation, the spin scalar product will coincide with $\overline{\Psi}\Phi$, where $\overline{\Psi} = \Psi^*\gamma^0$ is the adjoint spinor. Our definition without referring to Dirac matrices will clarify the relation between coordinate and gauge transformations in section 3.

The basic question is how we want to describe space-time. At the moment, position and time are merely parameters of the wave functions. This is the usual description in relativistic quantum mechanics. In the previous section, however, it was essential for the explanation of local gauge transformations that space corresponds to some operators on the wave functions. In order to generalize this "measurement principle" to the relativistic setting, we introduce "observables" for space and time as multiplication operators with the coordinate functions,

$$X^i \; : \; \Psi(x) \; \to \; x^i \, \Psi(x) \;\; , \qquad i = 0, \ldots, 3 \qquad . \tag{11}$$

We point out that these operators, especially the time operator X^0, are commonly not used in relativistic quantum mechanics. Nevertheless our description of Minkowski space makes sense, as we will see in the following.

The next step is the introduction of a scalar product on the wave functions. Usually one considers the positive product

$$(\Psi \mid \Phi) \; = \; \int_{\mathbb{R}^3} \Psi^*(t, \vec{x}) \, \Phi(t, \vec{x}) \, d\vec{x} \qquad , \tag{12}$$

where the spinors are integrated over a space-like hypersurface $t = $ const (we avoid the notation $\Psi^*\Phi = \overline{\Psi}\gamma^0\Phi$ with the adjoint spinor because we have no Dirac matrices at the moment). The integrand of $(\Psi \mid \Psi)$ can be interpreted as the probability density of the particle. For solutions of the Dirac equation, current conservation implies that (12) is independent of t. As an apparent problem of this scalar product, time measurements do not make sense, because expectations $(\Psi \mid X^0 \mid \Psi) = t \, (\Psi \mid \Psi)$ depend on the choice of the hypersurface. The only way out is to introduce a different scalar product where the spinors are also integrated over the time variable, namely

$$<\Psi \mid \Phi> \; = \; \int_{\mathbb{R}^4} \prec\Psi \mid \Phi\succ d^4x \qquad . \tag{13}$$

In contrast to (12), the scalar product (13) has no immediate physical interpretation. Nevertheless it causes no problem to consider (13) as the fundamental scalar product on the spinors, for the following reason: The scalar product does not enter in the physical equations (i.e. the Dirac equation and the classical field equations), we only need it for the probabilistic interpretation of the wave functions. For this final interpretation, we can just introduce (12) by an additional mathematical construction (see equation (27)).

We choose (13) as the fundamental scalar product on the wave functions. We denote the corresponding function space by $(H, <.|.>)$ and consider it together with the position/time operators (11) as an abstract scalar product space. In order to represent the vectors of H as wave functions, we choose an "eigenvector basis" $|x\alpha>$, $x \in \mathbb{R}^4$, $\alpha = 1, \ldots, 4$ of the position/time operators,

$$X^i \, |x\alpha> \; = \; x^i \, |x\alpha> \;\; , \qquad <x\alpha \mid y\beta> \; = \; s_\alpha \, \delta_{\alpha\beta} \, \delta^4(x - y) \qquad , \tag{14}$$

and define for $\Psi \in H$ the corresponding wave function by $\Psi^\alpha(x) = <x\alpha \mid \Psi>$. This is a relativistic analogue of (4). The additional factors $s_\alpha = \pm 1$ occur as a consequence of

the indefinite spin scalar product. The arbitrariness of the choice of $|x\alpha>$ leads to the transformations

$$
\begin{aligned}
|x\alpha> &\rightarrow \sum_{\beta=1}^{4} (U^{-1})_{\beta}^{\alpha}(x) \, |x\beta> \\
\Psi(x) &\rightarrow U(x) \, \Psi(x) \qquad \text{with} \qquad U(x) \in U(2,2) \qquad , \qquad (15)
\end{aligned}
$$

which we interpret as local $U(2,2)$ gauge transformations. According to this interpretation, we call $|x\alpha>$ a *gauge*. Notice that the gauge group is non-compact, which is directly related to the indefiniteness of the spin scalar product. Similar to the nonrelativistic case, the spectral measure is given by

$$
dE_x \; = \; \sum_{\beta=1}^{4} s_\beta \, |x\beta><x\beta| \, d^4x \qquad .
$$

We say that the spin dimension is $(2,2)$.

We remark that the indefiniteness of the scalar product $<.|.>$ leads to mathematical problems in the rigorous derivation of gauge transformations. A first, but not fully convincing attempt towards a satisfying mathematical formulation was made in [2] (which also contains a preliminary version of [3]).

3   The Free Dirac Operator, Lorentzian Transformations

Having introduced the Dirac wave functions in Minkowski space, we can now define the free Dirac operator: We fix the reference frame, choose a special gauge, and introduce the partial differential operator $i\partial\!\!\!/ := i\gamma^j \partial_j$ on the wave functions, where γ^j denote the usual (4×4)-matrices in the Dirac representation acting on the spinor index,

$$
\gamma^0 \; = \; \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad , \qquad \gamma^i \; = \; \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad , \quad i = 1,2,3
$$

$(\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices). Now we forget about the fact that the free Dirac operator was defined in a special gauge and a special reference frame and consider it as an operator on H. We denote this operator by G and write the free Dirac equation in the coordinate and gauge invariant form $(G - m)\Psi = 0$.

The Dirac operator is Hermitian (with respect to the scalar product $<.|.>$). This is most conveniently verified in the original gauge and reference frame where $G = i\partial\!\!\!/$: For simplicity, we rewrite the factors s_α in (10) with the matrix γ^0 and obtain

$$
\begin{aligned}
<G\Psi \,|\, \Phi> \; &= \; \int_{\mathbb{R}^4} \prec G\Psi \,|\, \Phi \succ_{|x} d^4x \; = \; \int_{\mathbb{R}^4} (i\gamma^j \partial_j \Psi)(x)^* \, \gamma^0 \, \Phi(x) \, d^4x \\
&= \; -i \int_{\mathbb{R}^4} (\partial_j \Psi)^* \, (\gamma^j)^* \, \gamma^0 \, \Phi \, d^4x \; = \; i \int_{\mathbb{R}^4} \Psi^* \, \gamma^0 \, (\gamma^0 \, (\gamma^j)^* \, \gamma^0) \, \partial_j \Phi \\
&= \; i \int_{\mathbb{R}^4} \Psi^* \, \gamma^0 \, \gamma^j \partial_j \Phi \; = \; \int_{\mathbb{R}^4} \prec \Psi \,|\, G\Phi \succ_{|x} d^4x \; = \; <\Psi \,|\, G\Phi> \qquad .
\end{aligned}
$$

Note that the γ-matrices are not Hermitian (more precisely, $(\gamma^0)^* = \gamma^0$ and $(\gamma^j)^* = -\gamma^j$ for $j = 1,2,3$), but they are self-adjoint with respect to the spin scalar product, $\prec \gamma^j \Psi \,|\, \Phi \succ = \prec \Psi \,|\, \gamma^j \Phi \succ$.

We point out that G is as operator on H a coordinate and gauge independent mathematical object (although its definition might depend on the special gauge and reference frame chosen at the beginning). Both coordinate and gauge transformations are merely passive transformations and lead to equivalent representations of G. Especially, transformations of the space-time coordinates and of the spinors are independent of each other. This is a major difference to the usual description of the Dirac equation. Therefore we will now discuss coordinate and gauge transformations and explain what "Lorentzian invariance" of the free Dirac operator exactly means in our setting.

We first look at the Dirac operator in a general gauge, but still in the original reference frame. The gauge transformation (15) leads to

$$i\partial\!\!\!/ \longrightarrow G = U\, i\partial\!\!\!/ \, U^{-1} = iG^j \frac{\partial}{\partial x^j} + B \qquad \text{with}$$
$$G^j(x) = U(x)\, \gamma^j\, U(x)^{-1} \;,\qquad B(x) = iU(x)\, \gamma^j\, (\partial_j U(x)^{-1}) \qquad .$$

As explained before, all these operators are equivalent; of course it is most convenient to work in the original gauge with $G = i\partial\!\!\!/$.

Next we look at a Lorentzian transformation. The transformation of the space-time variables is described by a coordinate transformation $x^j \to \Lambda^j_k x^k$, where Λ is an isometry of Minkowski space. The Dirac operator transforms like

$$i\partial\!\!\!/ \longrightarrow G = i\gamma^j \Lambda^k_j\, \partial_k \qquad . \tag{16}$$

Thus in the new reference frame, the Dirac operator no longer has the original form. This might seem to contradict Lorentzian invariance, but we must keep in mind that all operators related to each other by gauge transformations are equivalent. Thus we must look for a gauge transformation such that the Dirac operator in the new reference frame again coincides with $i\partial\!\!\!/$. As is shown in standard textbooks on relativistic quantum mechanics (see e.g. [1]), there is a (4×4)-matrix $U(\Lambda)$ with

$$U(\Lambda)\, \gamma^j \Lambda^k_j\, U(\Lambda)^{-1} = \gamma^k \qquad .$$

It is important for our purpose that this matrix is unitary with respect to the spin scalar product, $U(\Lambda) \in U(2,2)$. Therefore we can perform a gauge transformation with $U(x) \equiv U(\Lambda)$. In the new gauge we again have $G = i\partial\!\!\!/$, which shows Lorentzian invariance of G. To summarize, we describe a Lorentzian transformation in two steps. The first step is a coordinate, the second a gauge transformation. If both steps are performed at once, the transformation of the space-time coordinates and of the spinors are related in the usual way. Notice, however, that the second step is only a matter of convenience; we could just as well work in the new reference frame in any other gauge. This splitting of the space-time and spinor transformation becomes possible because the conditions $G^j = \gamma^j$ give a link between coordinate and gauge transformations.

We remark that it is not trivial that a spinorial equation can be described with passive coordinate and gauge transformations. As a counterexample, we consider the Weyl equation: Chiral fermions are usually described by two-component spinors χ satisfying the equation

$$i\sigma^j \frac{\partial}{\partial x^j}\, \chi = 0 \tag{17}$$

with Pauli matrices $\sigma^0 = \mathbb{1}$, $\vec{\sigma}$. This equation is Lorentzian invariant in the sense that there is a one-to-one correspondence between the solutions of (17) in different reference

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frames (see e.g. [1]). We assume that the Weyl operator $W := i\sigma^j \partial_j$ were given as a coordinate and gauge independent object (we will not specify the scalar product on the wave functions). After a Lorentzian transformation $x^j \to \Lambda^j_k x^k$, the Weyl operator has the form $W = i\sigma^k \Lambda^j_k \partial_j$. If W was Lorentzian invariant, it should be possible to perform a gauge transformation such that $W = i\sigma^j \partial_j$. Thus there should be a matrix $U(\Lambda)$ with

$$U(\Lambda)\, \sigma^k \Lambda^j_k\, U(\Lambda)^{-1} \;=\; \sigma^j \qquad .$$

Taking the trace on both sides, however, yields the equation $\Lambda^j_0 = \delta^j_0$, which is only satisfied for $\Lambda = \mathbb{1}$. We conclude that the Weyl operator cannot be described as a Lorentz invariant operator in our setting (this is not a serious problem because the Weyl equation can always be obtained as the left- or right-handed component of the massless Dirac equation).

4 General Definition of the Dirac Operator

In the previous section, we introduced the free Dirac operator as an operator on the wave functions in Minkowski space. We point out that this definition involved some global assumptions on space-time. First of all, the reference frames gave global coordinate systems. Furthermore, we chose a special gauge at the beginning and described Lorentzian transformations using global gauge transformations (i.e. transformations independent of x). According to the principle of general relativity, however, global conditions do not make sense. In this section, we replace them by corresponding local conditions, which will yield the general definition of the Dirac operator.

We start with the observation that the Minkowski metric can be derived from the Dirac matrices by $\eta^{ij} \mathbb{1} = \frac{1}{2} \{G^i, G^j\}$. This allows us to forget about the Minkowski metric and view Minkowski space simply as a four-dimensional vector space. For the generalization to curved coordinate systems, we replace this vector space by a four-dimensional (smooth) manifold M. Notice that M has no Lorentzian structure; we will later recover the Lorentzian metric from the Dirac operator.

Next we must generalize the scalar product space $(H, <.|.>)$ and the position/time operators to the setting of a space-time manifold. This can be most easily done with the help of spectral measures: We consider an indefinite scalar product space $(H, <.|.>)$ and introduce a spectral measure $(dE_x)_{x \in M}$ on M of spin dimension $(2,2)$. For any chart (x^i, U), we define corresponding position/time operators by

$$X^i \;=\; \int_U x^i\, dE_x \qquad . \tag{18}$$

We choose an "eigenvector basis" $|x\alpha>$, $x \in U$, $\alpha = 1, \ldots, 4$ of the X^i,

$$X^i\, |x\alpha> \;=\; x^i\, |x\alpha> \quad , \qquad <x\alpha\,|\,y\beta> \;=\; s_\alpha\, \delta_{\alpha\beta}\, \delta^4(x-y) \qquad , \tag{19}$$

and represent a vector $\Psi \in H$ as the wave function $\Psi^\alpha(x) = <x\alpha\,|\,\Psi>$, $x \in U$. This construction generalizes (9),(14) to curved coordinate systems. The spectral measure in (18) allows us to extend the usual changes of charts to the level of a calculus of operators.

After these preparations, we can introduce the Dirac operator G. First of all, it shall be a Hermitian operator on H. In a representation of H as wave functions, we assume it to be a partial differential operator of first order, i.e.

$$G \;=\; iG^j \frac{\partial}{\partial x^j} + B \tag{20}$$

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with (4×4)-matrices $G^j(x), B(x)$. Since G is (as an operator on H) a coordinate and gauge independent object, the matrices $G^j(x)$ and $B(x)$ have a well-defined behavior under coordinate and gauge transformations. As additional condition, we demand that the matrices G^j shall coincide locally with the usual Dirac matrices:

Def. 4.1 *A partial differential operator G on H is called* **Dirac operator***, if for any $p \in M$ there is a chart (x^i, U) around p and a gauge $|x\alpha>$ such that G has the form (20) with*

$$G^j(p) \;=\; \gamma^j \qquad . \tag{21}$$

This definition can also be understood in more physical terms: Let us assume that an observer (e.g. a spacecraft passing through the space-time point p) chooses a coordinate system and gauge satisfying (21). Since the Dirac matrices at p coincide with those of the free Dirac operator, the observer at p has the impression that space-time locally resembles Minkowski space. Thus we expect that the Dirac operator gives a Lorentz metric g_{jk}, which coincides with the Minkowski metric at p, $g_{jk}(p) = \eta_{jk}$. We can really derive this Lorentz metric from G; it is most conveniently given by

$$g^{jk}(x)\, \mathbb{1} \;=\; \frac{1}{2}\,\{G^j(x),\, G^k(x)\} \qquad . \tag{22}$$

In this way, our definition of the Dirac operator incorporates the principle of general relativity; the coordinate and gauge satisfying (21) gives a local reference frame. If the partial derivatives $\partial_j g_{kl}(p)$ of the metric also vanish (which can be arranged by choosing a so-called "normal coordinate system"), the reference frame is a "local system of inertia", where the observer feels no gravitational force at p. It might happen that $\partial_j G^k(p) \neq 0$ although $\partial_j g_{kl}(p)$ vanishes. This corresponds to a "spin-mixing force" which is related to our $U(2,2)$ symmetry. The matrix $B(x)$ can be used for the description of additional interactions, especially of electromagnetism $G = i\partial\!\!\!/ + e A\!\!\!/$. Note that $B(x)$ was not at all specified in our definition of the Dirac operator. This will turn out to be too general for physical applications. We postpone the discussion and specification of $B(x)$ to section 6.

Unfortunately, the equation (19) for the "eigenvector basis" is too simple and must be modified. This is only a minor technical point, but nevertheless we give it in detail: The δ^4-distribution in (19) is useful in combination with the integration measure d^4x, because integrations can be easily carried out using the relation $\int f(x)\, \delta^4(x)\, d^4x = f(0)$. On a manifold, however, one works with the invariant measure $d\mu := \sqrt{|g|}\, d^4x$ instead of d^4x. In order to compensate the factor $\sqrt{|g|}$ of the integration measure in the integral over the δ-distribution, we must view the combination $|g(y)|^{-\frac{1}{2}}\, \delta^4(x-y)$ as the "invariant δ-distribution" at a point y of the manifold. Thus we are led to replace (19) by

$$X^i\,|x\alpha> \;=\; x^i\,|x\alpha> \quad , \qquad <x\alpha\,|\,y\beta> \;=\; s_\alpha\,\delta_{\alpha\beta}\,\frac{1}{\sqrt{|g|}}\,\delta^4(x-y) \qquad . \tag{23}$$

This definition of a gauge is much better than (19) because $|x\alpha>$ now behaves under coordinate transformations $x \to x'(x)$ simply like $|x\alpha> \to |x'\alpha>$. We again introduce the wave functions by $\Psi^\alpha(x) = <x\alpha\,|\,\Psi>$ and obtain a representation of G as the differential operator (20). Notice that this indirect definition of a gauge by first introducing (19) and then replacing it by (23) was necessary because we had no invariant measure at the beginning. We must check that this procedure is consistent: The transition from (19) to (23) is described by the transformation $|x\alpha> \to |g(x)|^{-\frac{1}{4}}\,|x\alpha>$. Consequently,

the Dirac operator transforms as $G \rightarrow |g(x)|^{-\frac{1}{4}} G |g(x)|^{\frac{1}{4}}$, and thus $G^j \rightarrow G^j$, $B \rightarrow B + \frac{i}{4}G^j(\partial_j \log |g|)$. We conclude that the definition of the Dirac operator and the Lorentz metric (22) are independent of whether (19) or (23) are used.

We will in the following always work with (23) as the definition of a *gauge*. The freedom in the choice of the gauge again describes local $U(2,2)$ transformations of the wave functions

$$\Psi \rightarrow U(x)\,\Psi(x) \qquad \text{with} \qquad U(x) \in U(2,2) \qquad . \qquad (24)$$

We define the spin scalar product by (10). According to (23), the scalar product is obtained by integrating over the spin scalar product,

$$<\Psi \mid \Phi> \;=\; \int_M \prec\Psi \mid \Phi\succ d\mu \qquad . \qquad (25)$$

The Dirac equation is again introduced as the operator equation

$$(G - m)\,\Psi \;=\; 0 \qquad . \qquad (26)$$

On solutions of the Dirac equation, we finally define a generalization of the scalar product (12): We choose a space-like hypersurface $\mathcal{H}$ with normal vector field ν and set

$$(\Psi \mid \Phi)_{\mathcal{H}} \;=\; \int_{\mathcal{H}} \prec\Psi \mid G^j \mid \Phi\succ \nu_j \, d\mu_{\mathcal{H}} \qquad , \qquad (27)$$

where $d\mu_{\mathcal{H}}$ is the measure on $\mathcal{H}$ induced by the Lorentzian metric.

5 Construction of the Spin Derivative

We just introduced the Dirac operator on a manifold by combining the principle of general relativity with our $U(2,2)$ gauge symmetry. Although this definition was very natural, it is not clear what it precisely means: The Dirac operator is characterized by the matrices $G^j(x)$ and $B(x)$ in (20). We saw that $G^j(x)$ induces a Lorentzian structure and that $B(x)$ is composed of potentials which might be identified with usual gauge potentials. The precise physical interpretation of the degrees of freedom of $G^j(x),B(x)$ is not clear, however. Furthermore, we expect in analogy to the discussion of Lorentzian transformations in section 3 that coordinate and gauge transformations can be linked by imposing some conditions on the Dirac matrices. This would imply that the freedom in choosing the coordinate system is related to the $U(2,2)$ gauge symmetry. But this relation is also very vague at the moment.

In order to clarify the situation, we proceed in this section with some differential geometry. We will try to avoid abstract formalisms and prefer calculations in explicit coordinate systems and gauges. This is less elegant than a coordinate-free formulation, but it leads to a more elementary and direct approach. Our constructions are helpful for the physical interpretation, as will be explained in the next section 6. In non-technical terms, our aim is to rewrite the Dirac operator in a form which gives a better geometrical understanding of the matrices $G^j(x),B(x)$.

Before starting the mathematical analysis, we point out that we consider the Dirac operator (apart from the wave functions of the physical particles) as the only a-priori given object on the manifold. This means that all additional structure (like the metric,

curvature, classical potentials, and field tensors) must be constructed from G. It might seem unusual to take only the Dirac operator as the starting point of a classical gauge theory. The motivation for this very restrictive procedure is the following: A physical theory is more convincing if it is developed from few objects, which are given from the very beginning and are considered as the "fundamental" objects of the theory. In relativistic quantum mechanics, the Dirac operator is needed in any case for the formulation of the Dirac equation. The objects of classical field theory, however, either enter directly in the Dirac operator or can be expressed in terms of derivatives of potentials occurring in G. We conclude that it is both desirable and might be possible to consider them as derived objects, which are only constructed from the Dirac operator for a convenient formulation of the classical interactions.

The Lorentzian metric (22) induces the Levi-Civita connection ∇, which gives a parallel transport of vector and tensor fields along geodesics. The geometry of space-time can be described as usual with the Riemannian curvature tensor R^i_{jkl}, and we could thus formulate general relativity by writing down Einstein's field equations. This procedure only tells about part of the geometry, however. We did not at all use the matrix $B(x)$ and did not take into account the $U(2,2)$ gauge symmetry. If we want to understand all the degrees of freedom of the Dirac operator geometrically, it is a better idea to study the parallel transport of spinors (instead of tensor fields). In the infinitesimal version, this parallel transport is given by a $U(2,2)$ gauge covariant derivative D on the wave functions, which we call *spin derivative*. It is characterized by the condition that the derivative $D_j\Psi$ of a wave function shall again behave under gauge transformations according to (24). The basic question is if the Dirac operator induces a spin derivative.

For the construction of the spin derivative, we will analyze gauge transformations explicitly. In a special gauge and coordinate system, D shall have the representation

$$D_j \;=\; \frac{\partial}{\partial x^j} - iC_j(x) \tag{28}$$

with suitable (4×4)-matrices $C_j(x)$. Under a gauge transformation (24), it transforms like

$$D_j \;\rightarrow\; UD_jU^{-1} \;=\; \partial_j - iUC_jU^{-1} + U(\partial_jU^{-1}) \qquad . \tag{29}$$

Thus our aim is to find matrices $C_j(x)$ satisfying the transformation rule

$$C_j \;\rightarrow\; UC_jU^{-1} + iU(\partial_jU^{-1}) \qquad . \tag{30}$$

These matrices must be formed out of $G^j(x), B(x)$, which behave under gauge transformations like

$$G^j \;\rightarrow\; UG^jU^{-1} \;\;, \qquad B \;\rightarrow\; UBU^{-1} + iUG^j(\partial_jU^{-1}) \qquad . \tag{31}$$

It is far from being obvious how an explicit formula for the matrices C_j should look like. But we can already say something about its general structure: since only first derivatives of U occur in (30), we will only use G^j, B, and first derivatives of the Dirac matrices $\partial_k G^l$ for the construction. As a consequence, we need not care about second derivatives of U; they will not enter in our calculations. Constant gauge transformations (i.e. transformations U with $\partial_jU = 0$) are also irrelevant, because they only describe a common unitary transformation $. \rightarrow U \cdot U^{-1}$ of all matrices. The point of interest is the first order term $iU(\partial_jU^{-1})$ in (30).

We introduce some notation: In analogy to the bilinear and pseudoscalar covariants of the Dirac theory, we define the matrices

$$\sigma^{jk}(x) \;=\; \frac{i}{2}\,[G^j,\,G^k] \quad, \qquad \rho(x) \;=\; \frac{i}{4!}\,\epsilon_{jklm}\,G^j\,G^k\,G^l\,G^m \qquad\quad .$$

In a local reference frame (21), they coincide with the usual matrices σ^{jk},γ^5 in the Dirac representation (we use the notation ρ instead of γ^5, because the "tensor index" ^{5} might be confusing on a manifold). A (4×4)-matrix is called *even* or *odd* if it commutes resp. anti-commutes with ρ. The matrices

$$G^j \quad,\quad \rho G^j \quad,\quad \mathbb{1} \quad,\quad i\rho \quad,\quad \sigma^{jk} \tag{32}$$

form a basis of the 16-dimensional (real) vector space of self-adjoint matrices (with respect to $\prec.|.\succ$). G^j, ρG^j are odd; $\mathbb{1}$, $i\rho$, and σ^{jk} are even.

Since we must expect the matrices C_j to be complicated expressions in G^j and B, it is useful to first consider special gauges where these expressions should have a simple form. For this purpose, we study the term $\nabla_k G^j = \partial_k G^j + \Gamma^j_{kl}\,G^l$. Under coordinate transformations, it behaves like a tensor. Under gauge transformations, however, first derivatives of U occur,

$$\begin{aligned} \nabla_k G^j \;\to\; \nabla_k(UG^jU^{-1}) \;&=\; U(\nabla_k G^j)U^{-1} + (\partial_k U)G^jU^{-1} + UG^j(\partial_k U^{-1}) \\ &=\; U(\nabla_k G^j)U^{-1} - \Big[U(\partial_k U^{-1}),\,UG^jU^{-1}\Big] \qquad . \end{aligned} \tag{33}$$

We can use the second summand in (33) for a partial gauge fixing:

Lemma 5.1 *For any space-time point $p \in M$ there is a gauge such that*

$$\nabla_k G^j(p) \;=\; 0 \qquad . \tag{34}$$

Proof: Notice that the matrix $\partial_j \rho$ is odd, because

$$0 \;=\; \partial_j \mathbb{1} \;=\; \partial_j(\rho\rho) \;=\; (\partial_j\rho)\rho + \rho(\partial_j\rho) \qquad . \tag{35}$$

We start with an arbitrary gauge and construct the desired gauge with two subsequent gauge transformations:

1. According to (35), the matrix $i\rho(\partial_j\rho)$ is self-adjoint. We can thus perform a gauge transformation U with $U(p) = \mathbb{1}$, $\partial_j U(p) = \frac{1}{2}\rho(\partial_j\rho)$. The matrix $\partial_j\rho(p)$ vanishes in the new gauge, since

$$\partial_j\rho_{|p} \to \partial_j(U\rho U^{-1})_{|p} \;=\; \partial_j\rho_{|p} + \frac{1}{2}\,[\rho(\partial_j\rho),\,\rho]_{|p} \;=\; \partial_j\rho_{|p} - \rho^2(\partial_j\rho)_{|p} \;=\; 0 \qquad .$$

By differentiating the relation $\{\rho, G^j\} = 0$, we conclude that the matrix $\nabla_k G^j_{|p}$ is odd. We can thus represent it in the form

$$\nabla_k G^j_{|p} \;=\; \Lambda^j_{km}\,G^m_{|p} + \Theta^j_{km}\,\rho G^m \tag{36}$$

with suitable coefficients Λ^j_{km}, Θ^j_{km}.

This representation can be further simplified: According to Ricci's Lemma, we have at p

$$
\begin{aligned}
0 &= 2\nabla_n g^{jk} = \{\nabla_n G^j,\, G^k\} + \{G^j,\, \nabla_n G^k\} \\
&= 2\Lambda_{nm}^j\, g^{mk} - \Theta_{nm}^j\, 2i\rho\sigma^{mk} + 2\Lambda_{nm}^k\, g^{mj} - \Theta_{nm}^k\, 2i\rho\sigma^{mj} \quad,
\end{aligned}
\tag{37}
$$

and thus

$$
\Lambda_{nm}^j\, g_{|p}^{mk} = -\Lambda_{nm}^k\, g_{|p}^{mj} \qquad .
\tag{38}
$$

In the case $j = k \neq m$, (37) yields that $\Theta_{nm}^j = 0$. For $j \neq k$, we obtain $\Theta_{nj}^j\, \sigma^{jk} + \Theta_{nk}^k\, \sigma^{kj} = 0$ and thus $\Theta_{nj}^j = \Theta_{nk}^k$ (j and k denote fixed indices, no summation is performed). We conclude that there are coefficients Θ_k with

$$
\Theta_{km}^j = \Theta_k\, \delta_m^j \qquad .
\tag{39}
$$

2. We perform a gauge transformation U with $U(p) = \mathbb{1}$ and

$$
\partial_k U = -\frac{1}{2}\, \Theta_k\, \rho - \frac{i}{4}\, \Lambda_{kn}^m\, g^{nl}\, \sigma_{ml} \qquad .
$$

Using the representation (36) together with (38),(39), the matrix $\nabla_k G^j$ transforms like

$$
\begin{aligned}
\nabla_k G^j \;\to\; & \nabla_k G^j + [\partial_k U,\, G^j] \\
&= \Lambda_{km}^j\, G^m + \Theta_k\, \rho G^j - \Theta_k\, \rho G^j - \frac{i}{4}\, \Lambda_{kn}^m\, g^{nl}\, [\sigma_{ml},\, G^j] \\
&= \Lambda_{km}^j\, G^m - \frac{i}{4}\, \Lambda_{kn}^m\, g^{nl}\, 2i\, (G_m\, \delta_l^j - G_l\, \delta_m^j) \\
&= \Lambda_{km}^j\, G^m + \frac{1}{2}\, \Lambda_{kn}^m\, g^{nj}\, G_m - \frac{1}{2}\, \Lambda_{km}^j\, G^m = 0 \qquad . \qquad \blacksquare
\end{aligned}
$$

In general, the condition $\nabla_k G^j$ can only be satisfied "locally" in one point $p \in M$. We call a gauge satisfying (34) a *normal gauge* around p.

Now we look to which extent the gauge is fixed by Lemma 5.1. According to (33), a transformation between normal gauges must satisfy the condition $[U(\partial_j U^{-1}),\, U G^j U^{-1}]_{|p} = 0$. As a consequence, the matrix $iU(\partial_j U^{-1})_{|p}$ must be a multiple of the identity, as is verified in the basis (32) using the commutation relations between the Dirac matrices. Since constant gauge transformations and higher than first order derivatives of U are irrelevant, we can assume that $U(x)$ itself is a multiple of the identity. In other words, we can restrict ourselves to $U(1)$ gauge transformations. This is very helpful because in this special case there is a simple expression showing the transformation law (30), namely

$$
\begin{aligned}
\frac{1}{4}\, \mathrm{Re}\, \mathrm{Tr}(G_j\, B)\, \mathbb{1} \;\to\; & \frac{1}{4}\, \mathrm{Re}\, \mathrm{Tr}(G_j\, B)\, \mathbb{1} + \frac{1}{4}\, \mathrm{Re}\, \mathrm{Tr}\left(G_j\, G^k\, iU(\partial_j U^{-1})\, \mathbb{1}\right) \\
&= \frac{1}{4}\, \mathrm{Re}\, \mathrm{Tr}(G_j\, B)\, \mathbb{1} + iU(\partial_j U^{-1}) \qquad .
\end{aligned}
$$

We can identify this expression with C_j and use (28) as the definition for a spin derivative:

Def. 5.2 *We define the* **canonical spin derivative** D *by the condition that it has in normal gauges around p the form*

$$D_j(p) \;=\; \frac{\partial}{\partial x^j} \;-\; \frac{i}{4}\, Re\; Tr(G_j B)_{|p}\, \mathbb{1} \qquad . \tag{40}$$

In general gauges, the canonical spin derivative can be written as

$$D_j \;=\; \frac{\partial}{\partial x^j} \;-\; iE_j \;-\; \frac{i}{4}\, \mathrm{Re}\; \mathrm{Tr}(G_j B)\, \mathbb{1} \tag{41}$$

with additional matrices $E_j(x)$. These matrices are characterized by the condition that $E_j(p)$ vanishes in normal gauges around p and behaves according to (30) under $SU(2,2)$ gauge transformations. They are given by the formula

$$E_j \;=\; \frac{i}{2}\, \rho(\partial_j \rho) \;-\; \frac{i}{16}\, \mathrm{Tr}(G^m\, \nabla_j G^n)\, G_m G_n \;+\; \frac{i}{8}\, \mathrm{Tr}(\rho G_j\, \nabla_m G^m)\, \rho \qquad , \tag{42}$$

as can be verified by a straightforward calculation. Unfortunately, this expression is rather complicated. We will in the following always avoid working with E_j explicitly.

The canonical spin derivative has some nice properties. We first collect them in a theorem and discuss them afterwards.

Theorem 5.3 *The canonical spin derivative satisfies the equation*

$$[D_k, G^j] \;+\; \Gamma^j_{kl}\, G^l \;=\; 0 \qquad . \tag{43}$$

It is compatible with the spin scalar product,

$$\partial_j \prec\!\Psi \,|\, \Phi\!\succ \;=\; \prec\!D_j \Psi \,|\, \Phi\!\succ \;+\; \prec\!\Psi \,|\, D_j \Phi\!\succ \qquad . \tag{44}$$

The operator $iG^j D_j$ is a well-defined Hermitian operator on H. Furthermore,

$$-\,i\nabla_j \prec\!G^j \Psi \,|\, \Phi\!\succ \;=\; \prec\!iG^j D_j \Psi \,|\, \Phi\!\succ \;-\; \prec\!\Psi \,|\, iG^j D_j \Phi\!\succ \qquad . \tag{45}$$

Proof: The left side of (43) behaves under gauge transformations according to the adjoint representation $. \to U\,.\,U^{-1}$ of the gauge group. Thus it suffices to check (43) in a normal gauge, where

$$[D_k, G^j] + \Gamma^j_{kl}\, G^l \;=\; \nabla_k G^j \;-\; \frac{i}{4}\, \mathrm{Re}\; \mathrm{Tr}(G_j B)\, [\mathbb{1}, G^j] \;=\; 0 \qquad .$$

For the canonical spin derivative, the matrix C_j in (28) is self-adjoint, as can be verified with (41),(42). This immediately implies (44).

According to its behavior under coordinate and gauge transformations, we can view $iG^j D_j$ as an operator on H. Relation (44) yields the equation

$$\nabla_j \prec\!G^j \Psi \,|\, \Phi\!\succ \;=\; \prec\!D_j G^j \Psi \,|\, \Phi\!\succ \;+\; \prec\!G^j \Psi \,|\, D_j \Phi\!\succ \;+\; \Gamma^j_{jk} \prec\!G^k \Psi \,|\, \Phi\!\succ \qquad .$$

Applying (43) and the self-adjointness of the Dirac matrices, we obtain (45). If we integrate this equation with respect to the invariant measure $d\mu = \sqrt{|g|}\, d^4x$, the left side vanishes with Gauss' theorem. On the right side, we substitute (25) and conclude that the operator $iG^j D_j$ is Hermitian. ∎

Generally speaking, this theorem shows that the connections ∇, D, the Dirac matrices G^j, and the scalar products $g_{jk}, \prec.|.\succ$ can be consistently used in combination with each other, if all expressions are written in a coordinate and gauge invariant form (by appropriately using ∇_j, D_j, and Γ^j_{kl}). We can leave out the derivatives of G^j and may use the product rule inside the spin scalar product. Relation (44) formally corresponds to Ricci's Lemma in Riemannian geometry if we replace the metric by the spin scalar product and ∇ by D.

Since the operators G and $iG^j D_j$ are both Hermitian and coincide up to zero order contributions, we can represent the Dirac operator in the form

$$G \;=\; iG^j D_j + H \tag{46}$$

with a self-adjoint matrix $H(x)$. Under gauge transformations, H simply behaves according to the adjoint representation. This is an advantage over the formula (20), where the transformation of B involves derivatives of U.

Equation (45) implies a general version of current conservation: Assume that Ψ and Φ are eigenvectors of G with real eigenvalue m. Then the vector field $\prec G^j \Psi \mid \Phi \succ$ is divergence-free,

$$-i\nabla_j \prec G^j \Psi \mid \Phi \succ \;=\; \prec (G-H)\Psi \mid \Phi \succ \;-\; \prec \Psi \mid (G-H)\Phi \succ \;=\; 0 \qquad .$$

It is a natural generalization of the electromagnetic current $\overline{\Psi}\gamma^j \Psi$ of the Dirac theory. According to Gauß' law, the current conservation implies that the scalar product (27) is independent of the choice of the hypersurface $\mathcal{H}$. The integrand of the scalar product $(\Psi \mid \Psi)$ has the interpretation as the probability density of the particle.

The Christoffel symbol in (43) indicates that the spin derivative in some way includes the Levi-Civita connection ∇. The following bundle construction makes this relation more precise: We denote the sections of a vector bundle N over M by $\Gamma(N)$ (e.g. $\Gamma(TM)$ are the vector fields on M). The Levi-Civita connection is a connection on the tangent bundle TM,

$$\nabla \;:\; T_pM \times \Gamma(TM) \to T_pM \;:\; (X,Y) \to \nabla_X Y \qquad .$$

We can view the wave functions as sections of a vector bundle SM with fibre $(\mathbb{C}^4, \prec.|.\succ)$. In order to describe the $U(2,2)$ gauge symmetry, we make this *spin bundle* to a principal bundle with structure group $U(2,2)$, which acts on the fibre in the fundamental representation $. \to U.$. The spin derivative is a connection on the spin bundle,

$$D \;:\; T_pM \times \Gamma(SM) \to S_pM \;:\; (X,\Psi) \to D_X \Psi \qquad .$$

We define L_pM as the vector space of self-adjoint transformations of S_pM. In a special gauge, L_pM can be represented as (4×4)-matrices; (32) gives an explicit basis. The gauge transformations act on the matrices according to the adjoint representation $. \to U_{|p} . U_{|p}^{-1}$. We denote the corresponding principal bundle by LM. The spin derivative induces a connection on LM by

$$D \;:\; T_pM \times \Gamma(LM) \to L_pM \;:\; (X,A) \to [D_X, A] \qquad . \tag{47}$$

We can view the tangent bundle as a subspace of LM, because there are canonical embeddings and projections

$$\iota_p \;:\; T_pM \hookrightarrow L_pM \;:\; X \to X^i\, G_i$$

$$\pi_p \;:\; L_pM \to T_pM \;:\; A \to \frac{1}{4}\,\mathrm{Tr}(G^j A)\,\frac{\partial}{\partial x^j}$$

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with $\pi\iota = \mathrm{id}$. By restricting the connection (47) to the tangent bundle, we can form a connection $\tilde{\nabla}$ on TM,

$$\tilde{\nabla} \ : \ T_pM \times \mathcal{S}(TM) \to T_pM \ : \ (X,Y) \to \pi\,[D_X, \, \iota Y] \qquad .$$

Equation (43) states that this connection coincides with the Levi-Civita connection, since

$$\begin{aligned}
[D_X, \, \iota Y] \ &= \ X^j\,[D_j, \, G^k\,Y_k] \ = \ X^j\left((\partial_j Y_k)\,G^k \ - \ \Gamma_{jl}^k\,Y_k\,G^l\right) \\
&= \ X^j\,(\nabla_j Y_k)\,G^k \ = \ \iota\,\nabla_X Y
\end{aligned} \qquad (48)$$

and thus

$$\tilde{\nabla}_X Y \ = \ \pi\,[D_X, \, \iota Y] \ = \ \pi\iota\,\nabla_X Y \ = \ \nabla_X Y \qquad .$$

In this way, the Levi-Civita connection is recovered as a suitable restriction of the spin connection. The geometry of the spin bundle generalizes the Lorentzian geometry of the manifold.

It remains to construct the geometrical invariants of the spin connection. On an elementary level, one can look for expressions in D_k, G^j which transform both like a tensor and according to a local representation of the gauge group. This leads to the definition of the 2-forms

$$\begin{aligned}
T_{jk} \ &= \ \frac{i}{2}\,([D_j, G_k] - [D_k, G_j]) \\
R_{jk} \ &= \ \frac{i}{2}\,[D_j, D_k] \qquad ,
\end{aligned}$$

which are called *torsion* and *curvature*.

Theorem 5.4 *The canonical spin connection is torsion-free. Curvature has the form*

$$R_{jk} \ = \ \frac{1}{8}\,R_{mnjk}\,\sigma^{mn} \ + \ \frac{1}{2}\,(\partial_j a_k - \partial_k a_j)\,\mathbb{1} \qquad (49)$$

with the Riemannian curvature tensor R_{mnjk} and the potential

$$a_j \ = \ \frac{1}{4}\,Re\ Tr(G_j B) \qquad . \qquad (50)$$

Proof: Relation (43) yields

$$[D_j, \, G_k] \ = \ [D_j, \, g_{kl}\,G^l] \ = \ (\partial_j g_{kl})\,G^l \ - \ g_{kl}\,\Gamma_{jm}^l\,G^m \ = \ \Gamma_{jk}^m\,G_m$$

and thus, using that the Levi-Civita connection is torsion-free,

$$T_{jk} \ = \ \frac{i}{2}\,(\Gamma_{jk}^m - \Gamma_{kj}^m)\,G_m \ = \ 0 \qquad .$$

We iterate formula (48) and express the Riemannian curvature tensor in terms of the curvature of the spin connection,

$$\begin{aligned}
G_i\,R_{jkl}^i\,Z^l \ &= \ \iota\,(\nabla_j\nabla_k Z - \nabla_k\nabla_j Z) \\
&= \ [D_j, \, [D_k, \, \iota Z]] - [D_k, \, [D_j, \, \iota Z]] \\
&= \ [[D_j, \, D_k], \, \iota Z] \ = \ -2i\,[R_{jk}, \, \iota Z] \qquad .
\end{aligned}$$

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This equation determines curvature up to a multiple of the identity matrix,

$$R_{jk}(x) \;=\; \frac{1}{8}\, R_{mnjk}\, \sigma^{mn} \;+\; \lambda_{jk}\mathbf{1} \qquad .$$

Thus it remains to calculate the trace of curvature,

$$\frac{1}{4}\,\mathrm{Tr}(R_{jk}) \;=\; \frac{1}{8}\,\mathrm{Tr}(\partial_j C_k - \partial_k C_j) \;=\; \frac{1}{2}\,(\partial_j a_k - \partial_k a_j) \qquad ,$$

where we used (41) and the fact that the matrices E_j are trace-free. ∎

We remark that the canonical spin derivative is not the only spin derivative which can be constructed from the Dirac operator. A general spin derivative $\tilde{D}$ differs from the canonical spin derivative by matrices $F_j = \tilde{D}_j - D_j$ which transform according to the adjoint representation of the gauge group. It is no loss of generality to work with the canonical spin derivative because the matrices F_j can also be dealt as additional potentials. Because of the useful properties of Theorem 5.4, it is most convenient to use the canonical spin derivative.

6 Interpretation, the Physical Dirac Operator

For the physical interpretation of the previous constructions, we return to the discussion of the moving observer on page 8. We saw that the coordinate system and gauge satisfying (21) give a local reference frame for the observer. The gravitational field can be compensated locally by choosing a normal coordinate system. According to Lemma 5.1, we can further compensate the dynamics of the Dirac matrices: in a normal gauge even the derivatives of the Dirac matrices vanish, $\partial_k G^j(p) = \nabla_k G^j(p) = 0$. This means that the above mentioned "spin mixing force" can also be compensated locally by choosing an appropriate coordinate system and gauge. We cannot do even better because second order derivatives of the metric and the Dirac matrices involve the Riemannian curvature tensor. We call a coordinate system and gauge satisfying the conditions $G^j(p) = \gamma^j$, $\partial_k G^j(p) = 0$ a *normal reference frame* around p.

We come to the discussion of the matrix $B(x)$, which was not at all specified in Def. 4.1. The strongest local condition is to impose that $B(p)$ vanishes in a suitable normal reference frame around p. This condition turns out to be of physical interest, and we use it as the definition for the physical Dirac operator:

Def. 6.1 *A Dirac operator G is called* **physical Dirac operator** *if for any $p \in M$ there is a normal reference frame around p such that $B(p) = 0$.*

In other words, the physical Dirac operator is characterized by the condition that it locally coincides with the free Dirac operator, which means more precisely that there is a coordinate system and a gauge with $G^j(p) = \gamma^j$, $\partial_k G^j(p) = 0$ and $B(p) = 0$.

We again consider the physical Dirac operator as the only a-priori given object on the manifold and construct the metric and the canonical spin derivative from G. In a normal reference frame with $G = iG^j\partial_j$, the canonical spin derivative (40) coincides with the partial derivatives, $D_j = \partial_j$. Thus the physical Dirac operator has the representation

$$G \;=\; iG^j D_j \qquad , \tag{51}$$

which is also valid in a general gauge and coordinate system. In comparison to the representation (20), the degrees of freedom of the matrix $B(x)$ now occur in the canonical spin derivative. This has the advantage that they can be immediately identified with the $U(2,2)$ gauge potentials in (41). The last summand in (41) describes the $U(1)$ phase transformations of electrodynamics. We define the electromagnetic potential by

$$A_j \;=\; \frac{1}{4e}\,\mathrm{Re}\,\mathrm{Tr}(G_j\,B)$$

and rewrite the canonical spin derivative in the more familiar form

$$D_j \;=\; \frac{\partial}{\partial x^j} \;-\; iE_j \;-\; ieA_j \qquad .$$

The matrices E_j take into account the gravitational field. According to Theorem 5.4, the curvature of the spin connection is formed of the Riemannian curvature tensor and the electromagnetic field tensor $F_{jk} = \partial_j A_k - \partial_k A_j$. We can write down the classical action in terms of these tensor fields. The classical variational principle yields a unified description of general relativity and electromagnetism as a $U(2,2)$ gauge symmetry.

The definition of the physical Dirac operator can also be understood in more technical terms with the canonical spin derivative: The Dirac operator of Def. 4.1 has the representation (46) with a self-adjoint matrix $H(x)$. According to the construction of the spin derivative, the matrices $G^j H$ are trace-free. Thus $H(x)$ is characterized by twelve degrees of freedom, which correspond to additional potentials (more precisely, 1 scalar, 1 pseudoscalar, 4 pseudovector, and 6 bilinear potentials). These potentials are not gauge potentials. They do not seem to occur in nature. Therefore it is a reasonable physical condition to assume that H vanishes. This leads to the representation (51) of the physical Dirac operator.

Our definition of the Dirac operator introduces the gauge potentials and in this way replaces the usual minimal coupling procedure. We explain for clarity why simple minimal coupling does not make sense in the context of our $U(2,2)$ symmetry: Assume that we had (instead of constructing the spin derivative from the Dirac operator) introduced a $U(2,2)$ gauge covariant derivative D of the form (28) with gauge potentials C_j. According to minimal coupling, we must replace the partial derivatives in the free Dirac operator by gauge covariant derivatives, which gives the operator $i\gamma^j D_j$. This operator does not make sense, however. First of all, it is not Hermitian (with respect to $<.|.>$). Furthermore, the current $\prec\gamma^j\Psi\,|\,\Psi\succ$ is in general not divergence-free, so that the probabilistic interpretation of the Dirac wave function breaks down. The basic reason why minimal coupling does not work is that the $U(2,2)$ gauge potentials C_j do in general not commute with the Dirac matrices γ^j. In order to bypass these problems, we must replace the γ-matrices by dynamical Dirac matrices $G^j(x)$. The two replacements $\partial_j \to D_j$ and $\gamma^j \to G^j$ must be coordinated in such a way that the Dirac operator $iG^j D_j$ is Hermitian and allows a reasonable definition of a conserved current. This is accomplished by our definition of the Dirac operator and the subsequent construction of the spin derivative.

Our description of gravitation differs considerably from the usual formulation of general relativity as a gauge theory (see e.g. [5]). As one of the major differences, we avoid the principal bundle of orthonormal frames (with the Lorentz group as structure group); instead we are working with the spin bundle and local $U(2,2)$ transformations. This replacement is possible because the subgroup of $U(2,2)$ generated by the bilinear covariants σ^{ij} is locally isomorphic to the Lorentz group. In contrast to the connection on the bundle

of orthonormal frames, the canonical spin connection is torsion-free. Remember that the $U(2,2)$ spin bundle arose as a consequence of our measurement principle for space and time. Thus, as an advantage of our procedure, the usual bundle constructions for the definition of gauge fields and of spinors in curved space-time are no longer necessary.

7 Conclusion

We saw that the adaptation of the ideas in [3] to the relativistic context yields a local $U(2,2)$ gauge symmetry of the Dirac equation. In order to describe the physical interactions with this gauge symmetry, it is necessary to consider the Dirac operator as the basic object on the manifold. The gauge potentials are implicitly contained in the Dirac operator. By constructing the spin derivative, they are recovered as describing the electromagnetic and gravitational field. In this way, we conclude that the local $U(2,2)$ symmetry in relativistic quantum mechanics makes physical sense. The concept of measurability of space-time gives a fundamental explanation for this gauge symmetry.

Our description has the advantage that both the Dirac theory and classical field theory are developed from few a-priori given objects: The fermionic particles correspond to vectors of the indefinite scalar product space $(H, <.|.>)$. Space-time is described by the spectral measure dE_x on the manifold M. The Dirac operator gives the gauge potentials and determines the interaction between the fermions and the gauge fields. This description is conceptually simple. It is the starting point for further constructions which finally lead to the "Principle of the Fermionic Projector" as introduced in [4].

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