Spectral Networks and G_2

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based on joint work with Andy Neitzke

Nonabelian Hodge Correspondence

Let us fix:

- a (compact) Riemann surface C;
- a complex reductive Lie group G (e.g. $\mathrm{GL}_n, SL_n, \mathrm{G}_2$).

To this data one can associate three different moduli spaces:

1. $\mathcal{M}_\mathrm{H} = \mathcal{M}_\mathrm{H}(G, C)$ - the moduli space of (stable) G-**Higgs bundles**;
2. $\mathcal{M}_\mathrm{dR} = \mathcal{M}_\mathrm{dR}(G, C)$ - the moduli space of (irreducible) **flat G-bundles**;
3. $\mathcal{M}_\mathrm{B} = \mathcal{M}_\mathrm{B}(G, C)$ - the **character variety** of representations $\mathrm{Hom}(\pi_1 C, G)$.

Nonabelian Hodge Correspondence

To this data one can associate three different moduli spaces:

1. $\mathcal{M}_{\mathrm{H}}$ - the moduli space of (polystable) G-**Higgs bundles**;
2. $\mathcal{M}_{\mathrm{dR}}$ - the moduli space of (irreducible) **flat G-bundles**;
3. $\mathcal{M}_{\mathrm{B}}$ - the **character variety** of representations $\mathrm{Hom}(\pi_1 C, G)$.

These spaces are all diffeomorphic:

$\mathcal{M}_{\mathrm{dR}} \to \mathcal{M}_{\mathrm{B}}$ by taking holonomies;

$\mathcal{M}_{\mathrm{B}} \to \mathcal{M}_{\mathrm{dR}}$ by solving a Riemann-Hilbert problem.

Nonabelian Hodge Correspondence

To this data one can associate three different moduli spaces:

1. $\mathcal{M}_{\mathrm{H}}$ - the moduli space of (polystable) ***G*-Higgs bundles**;
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The **Nonabelian Hodge Correspondence** asserts that there is a $\mathbb{C}^*$-family of diffeomorphisms

$$\mathrm{NHC}^{\zeta} : \mathcal{M}_{\mathrm{H}} \to \mathcal{M}_{\mathrm{dR}}$$

obtained by solving **Hitchin's equations** - a difficult PDE.
Part of the motivation for this work is to describe these diffeomorphisms more explicitly.

Nonabelianization of Higgs bundles

The standard treatment of Higgs bundles inherently **abelianizes** them. Recall

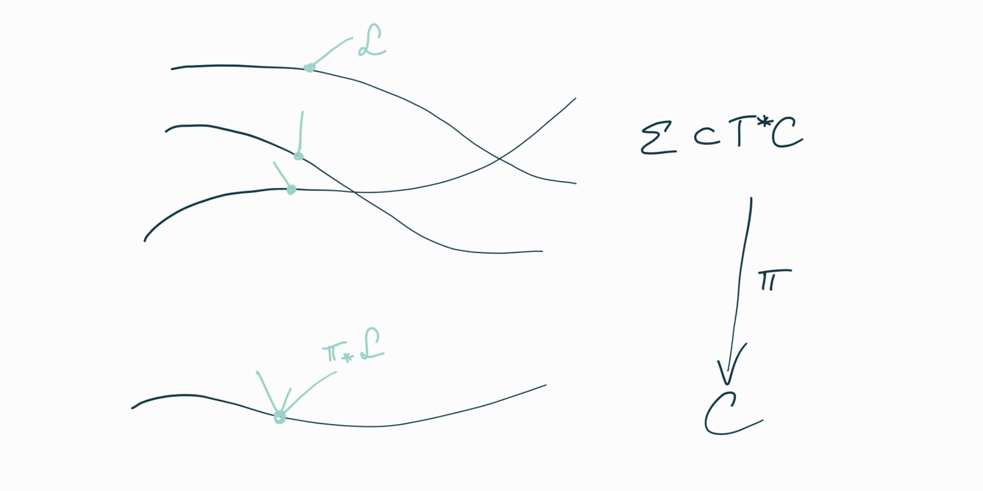
$$\mathcal{M}_{\mathrm{H}} = \left\{(P,\varphi) : P \text{ principal } G-\text{bundle}, \varphi \in H^0(C, \mathrm{ad}\, P \otimes K_C\right\} / \sim .$$

There is also a *vector bundle version*: E.g. for $\mathrm{GL}_n(\mathbb{C})$ or $\mathrm{SL}_n(\mathbb{C})$ one considers *holomorphic* vector bundles $\mathcal{E}$ of rank n and $\varphi : \mathcal{E} \to \mathcal{E} \otimes K_C$.

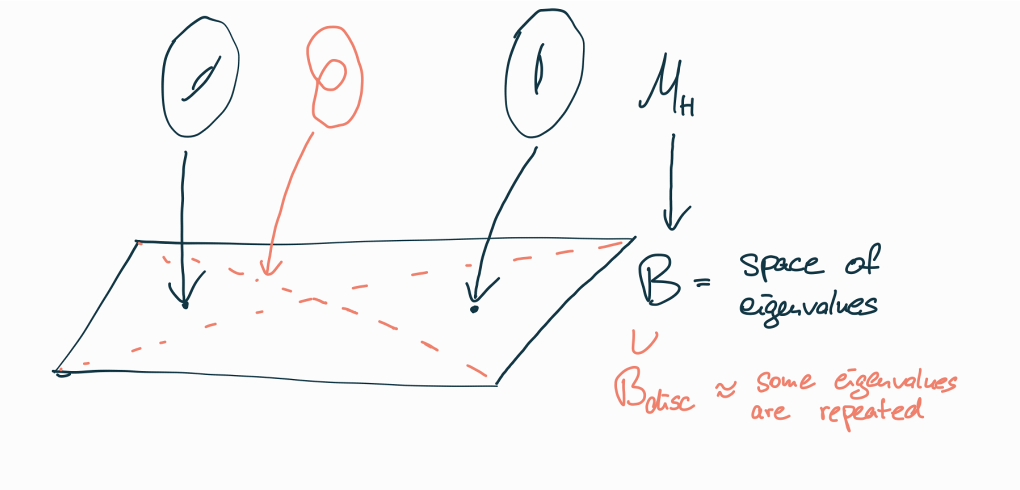
Abelianization associates to this the **spectral curve** $\Sigma \subset \mathrm{Tot}(K_C)$ of eigenvalues of φ and a line bundle $\mathcal{L} \to \Sigma$ of *eigenvectors*, i.e. such that

$$\pi_* \mathcal{L} = \mathcal{E}.$$

The spectral correspondence



The Hitchin integrable system



Nonabelianization of flat connections

One could try the same for flat G-bundles: Take a line bundle L on the cover $\pi : \Sigma \to C$ together with an abelian (i.e. $\mathbb{C}^*$-) connection ∇^{ab}. Then *hope* that the pushforward $\pi_*(L, \nabla^{ab})$ is a flat bundle on C.

This works well *away* from **ramification points** r of π. But around r, there is necessarily nontrivial monodromy and the construction needs to be modified. A **spectral network** captures the combinatorics of these modifications.

All of the above holds for **non-compact surfaces** with the appropriate definitions/modifications. Indeed, this is the setup I will describe.

Outline

So here's the plan:

1. Review Higgs bundles and their relation to flat connections, Stokes phenomena, cluster varieties etc. for $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{SL}_3(\mathbb{C})$;
2. Explain nonabelianization in this context;
3. Describe current progress for G_2.

The setup

The "easiest" Higgs bundles are those in the **Hitchin section** $\mathcal{H}$:

$$\mathcal{E} = K_C^{1/2} \oplus K_C^{-1/2}, \ \varphi = \begin{pmatrix} 0 & q_2 \\ 1 & 0 \end{pmatrix},$$

where $1 \in \mathrm{Hom}(K_C^{1/2}, K_C^{-1/2} \otimes K_C) \simeq \mathcal{O}$ and $q_2 \in H^0(C, K_C^2)$.

Recall that there are also diffeomorphisms

- NHC[1] : $\mathcal{H} \to \mathrm{Teich}(C)$, the Teichmüller space of C, and
- the *conformal limit* $\mathcal{CL}_\hbar : \mathcal{H} \to \mathrm{Op}_C$, the space of *opers*.

SL$_2$(ℂ)-opers

Opers are global versions of the Schrödinger equation

$$\left[-\hbar^2\partial_z^2 + P_2(z)\right]\psi(z) = 0.$$

Converting it into a linear differential operator yields a rank 2 vector bundle that is holomorphically

$$0 \to K_C^{1/2} \to E_\hbar \to K_C^{-1/2} \to 0$$

with connection (in a distinguished trivialization)

$$\nabla_{\hbar,q_2} = d + \hbar^{-1}\begin{pmatrix} 0 & q_2 \\ 1 & 0 \end{pmatrix} dz.$$

Example: The Airy function

The easiest example is given by the **Airy equation**

$$\psi'' + z\psi = 0$$

in the complex plane. In its two-dimensional space of entire solutions is a line of solutions, spanned by the **Airy function** $\mathrm{Ai}(z)$, distinguished by

$$\lim_{z \to \infty^+} \mathrm{Ai}(z) = 0.$$

Asymptotically, for $|\arg(z)| < 2\pi/3$, around $z = \infty$,

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi} z^{1/4}} \exp\left(-\frac{2}{3} z^{3/2}\right) \left(1 - \frac{5}{48 z^{3/2}} + \frac{385}{4608 z^3} + \dots\right).$$

The Airy function

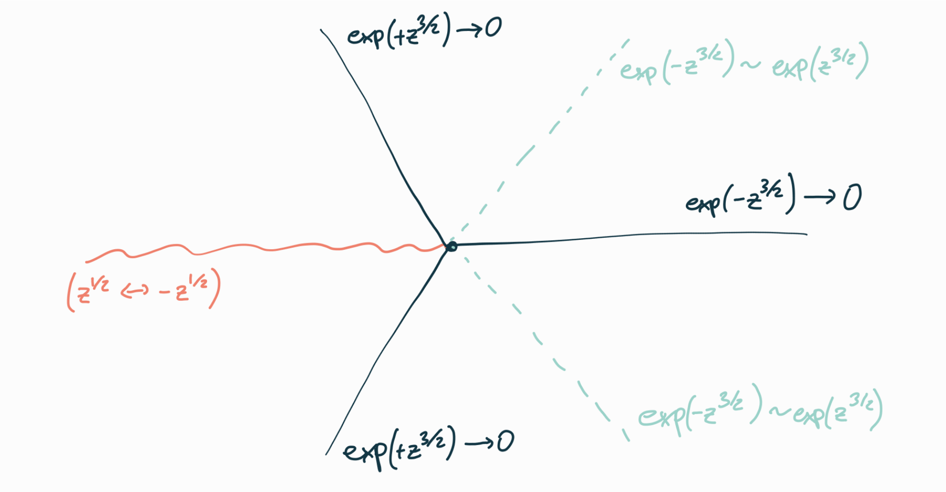
$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi} z^{1/4}} \exp\left(-\frac{2}{3} z^{3/2}\right)\left(1 - \frac{5}{48 z^{3/2}} + \frac{385}{4608 z^3} + \dots\right).$$

Meanwhile, all other solutions obey the following, including the distinguished *Bairy functon* Bi(z):

$$\psi(z) \sim \exp\left(+\frac{2}{3} z^{3/2}\right)$$

as $z \to \infty^+$. So once $\arg(z) > \pi/3$, the Airy function seizes to be the exponentially declining solution. This is known as the **Stokes phenomenon**.

A simple turning point



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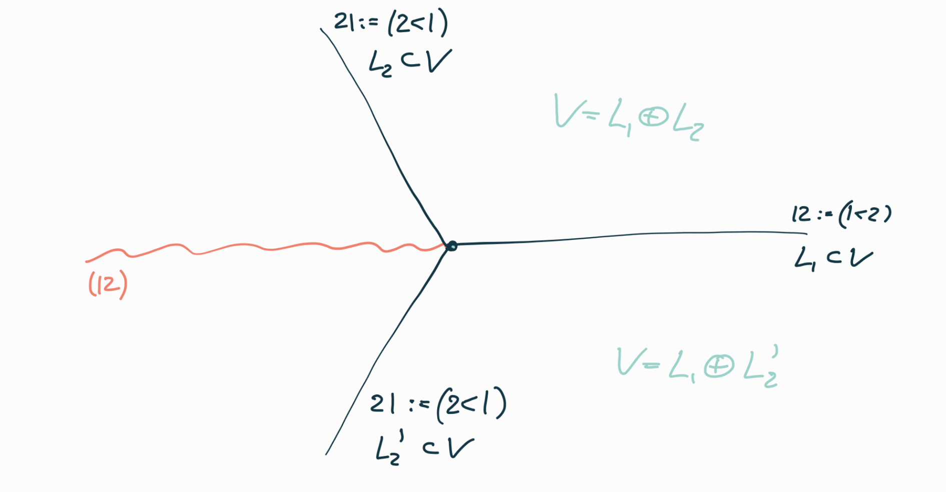
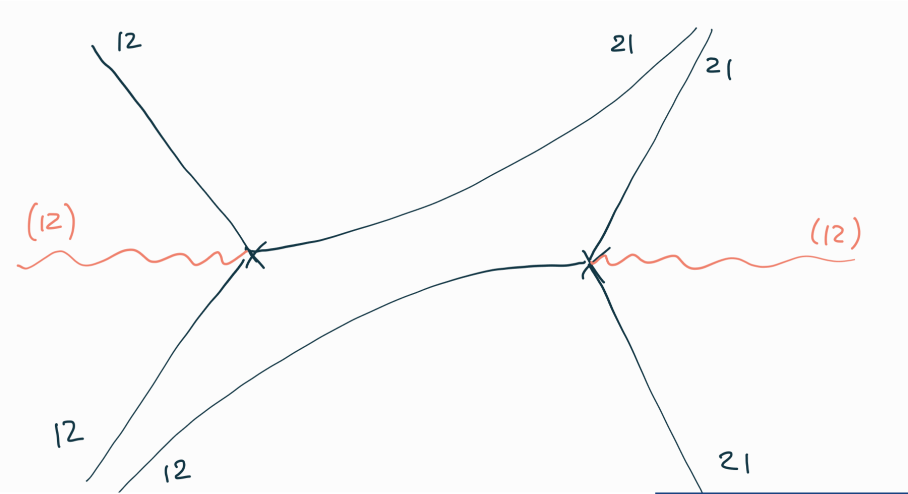
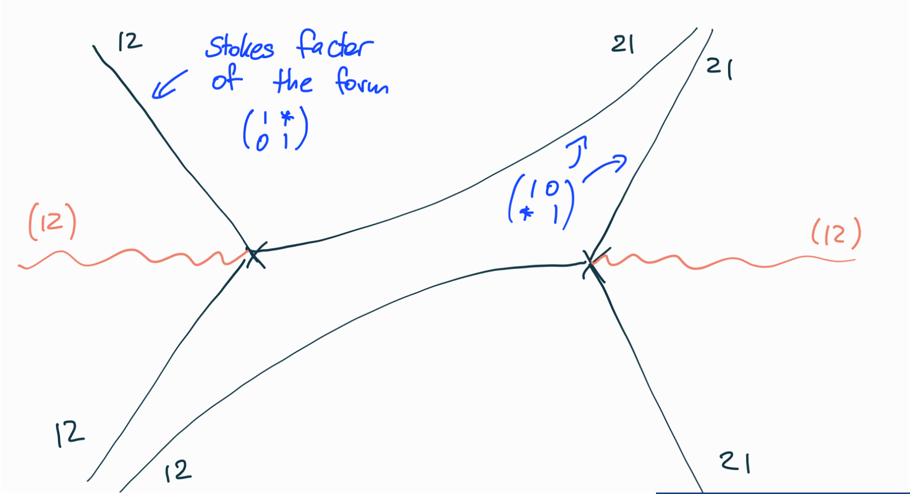


Two turning points



Repairing modifications



Cluster coordinates

Given a polynomial P_2 of degree k, the space of solutions is parametrized by the $(k+2)$ asymptotic lines $L_1, \ldots, L_{k+2} \in \mathbb{CP}^1$ up at simultaneous action by $\mathrm{SL}_2(\mathbb{C})$, hence by a collection of cross-ratios.

In fact, there is some extra reality hidden in this. The Hitchin equations on $\mathcal{H}$ reduce to studying harmonic maps

$$g : \mathbb{C} \to \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2 \simeq \mathbb{D}^2.$$

The asymptotic geometry is governed by (convex) $(k+2)$-gons in the boundary $\mathbb{RP}^1$ [Han-Tam-Treibergs-Wan, Fock-Goncharov, Gaiotto-Moore-Neitzke].

Setup

For $\mathrm{SL}_3(\mathbb{C})$ the Hitchin section is given by

$$\mathcal{E} = K_C \oplus \mathcal{O} \oplus K_C^{-1}, \varphi = \begin{pmatrix} 0 & q_2 & q_3 \\ 1 & 0 & q_2 \\ 0 & 1 & 0 \end{pmatrix},$$

$$q_2 \in H^0(C, K_C^2),\ q_3 \in H^0(C, K_C^3).$$

We will mostly be interested in the case $q_2 = 0$ because then the harmonic metric becomes diagonal. For $C = \mathbb{C}$ and $q_3 = P_3(z)dz^3$, the Hitchin equations reduce to studying harmonic maps

$$g : \mathbb{C} \to \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_3 \simeq \mathbb{H}^3.$$

Convex Polygons

Let

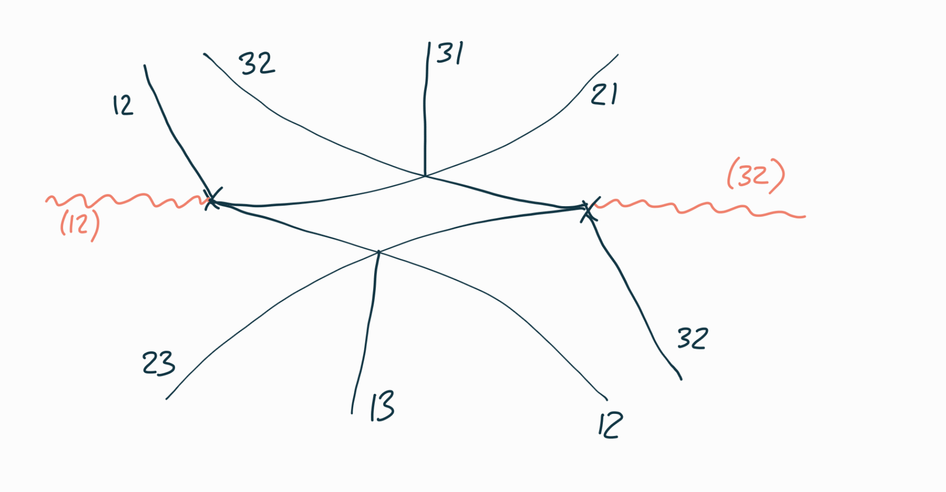
$$\mathcal{MC}_k := \left\{ q_3 = P_3(z)dz^3 : P_3 \text{ is a polynomial of degree } k \right\} / \operatorname{Aut}(\mathbb{C}),$$

$$\mathcal{MP}_n := \left\{ \text{convex } n-\text{gons in } \mathbb{RP}^2 \right\} / \mathrm{SL}_3(\mathbb{R}).$$

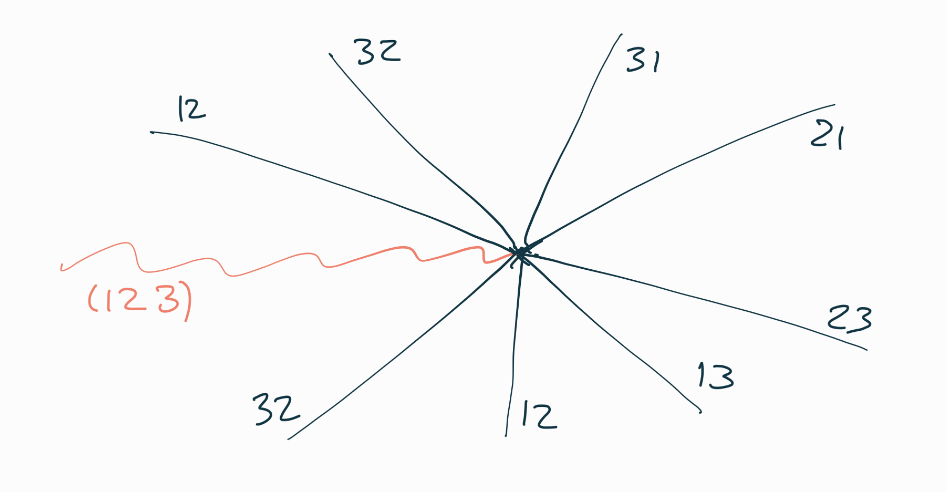
Theorem [Dumas-Wolf '14]

There is a homeomorphism $\mathcal{MC}_k \to \mathcal{MP}_{k+3}$.

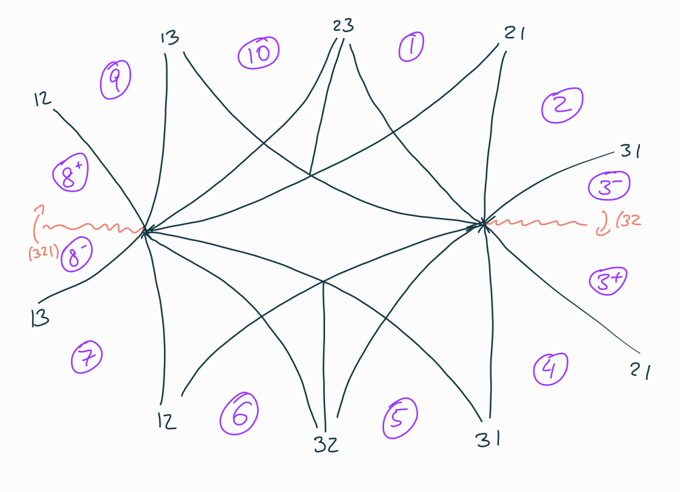
A new phenomenon



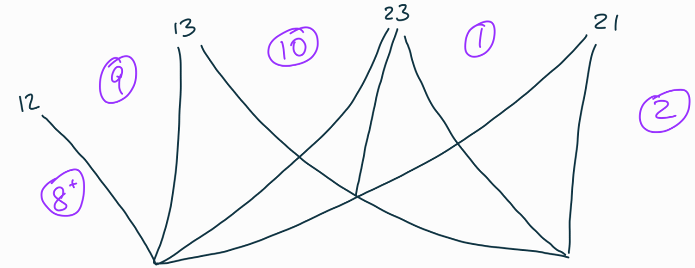
Simple zero of a cubic differential



Two simple zeroes of a cubic differential



Line decompositions and $\mathrm{Gr}_3(5)$



$$\mathcal{L}_1^{(8)}\oplus\mathcal{L}_2^{(8)} \subset \mathcal{L}_1^{(9)} \supset \mathcal{L}_1^{(10)}\oplus\mathcal{L}_3^{(10)}$$

$$\mathcal{L}_2^{(8)} = \mathcal{L}_2^{(9)} = \mathcal{L}_2^{(10)}$$

$$\mathcal{L}_3^{(8)} = \mathcal{L}_3^{(8)} = \mathcal{L}_3^{(10)}$$

Genericity

$$\Rightarrow \mathcal{L}_1^{(9)} = \left(\mathcal{L}_1^{(8)}\oplus\mathcal{L}_2^{(8)}\right)\cap\left(\mathcal{L}_1^{(10)}\oplus\mathcal{L}_3^{(10)}\right)$$

Setup

For $\mathrm{G}_2(\mathbb{C})$ the Hitchin section is given by

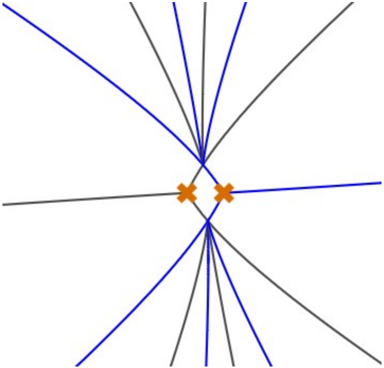
$$\mathcal{E} = K_C^3 \oplus K_C^2 \oplus K_C \oplus \mathcal{O} \oplus K_C^{-1} \oplus K_C^{-2} \oplus K_C^{-3}, \varphi = \varphi(q_2, q_6),$$
$$q_2 \in H^0(C, K_C^2),\ q_3 \in H^0(C, K_C^3).$$

Again, we are interested mostly in the case $q_2 = 0$. For $C = \mathbb{C}$ and $q_6 = P_6(z)dz^6$, the Hitchin equations reduce to studying harmonic maps

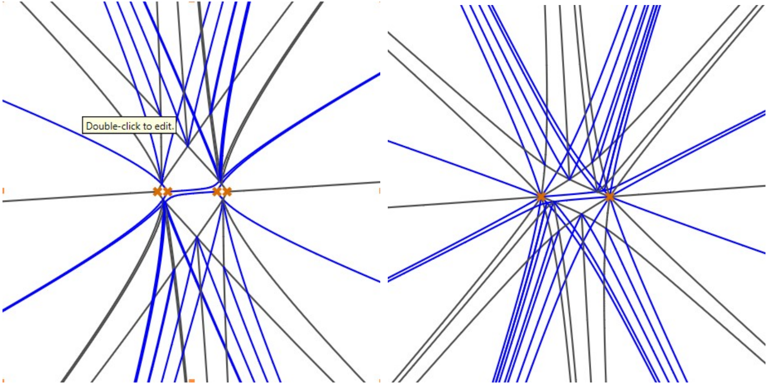
$$g : \mathbb{C} \to {\mathrm{G}_2}'/\mathrm{SU}_3,$$

where ${\mathrm{G}_2}'$ is the split real form of $\mathrm{G}_2(\mathbb{C})$.

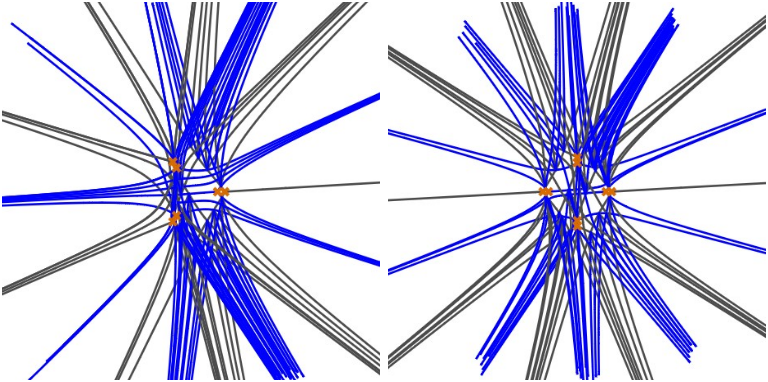
G_2 with q_2 turned on



G_2 Spectral Networks



G_2 Spectral Networks



So what does the asymptotic geometry look like?
An **annihilator polygon** is a cyclically ordered set $S = (x_1, \ldots, x_p)$ of points $x_i \in \mathrm{Ein}^{2,3}$, the projectivized light cone in $\mathbb{R}^{3,4}$, such that $(x_i, x_{i+1}) = 0$[1] and

$$\mathrm{Ann}(x_i) := \{y \in \mathrm{Im}\,\mathbb{O}' : x_i y = 0\} = x_{i-1} \oplus x_i \oplus x_{i+1}. \tag{1}$$

Example: The weight space decomposition for $\mathrm{G_2}'$ gives an annihilator hexagon.

Theorem [Evans '22]

Given a sextic differential $q_6 = P_6 dz^6$, where P_6 is a polynomials of degree k, the harmonic map construction produces an annihilator $k+6$-gon.

Problem: Neither injectivity nor surjectivity of this map is known.

[1] In particular, $ax_i + bx_{i+1}$ is null for all $a, b \in \mathbb{R}$.

We [Neitzke-S.] construct cluster coordinates $\mathcal{X}_\gamma$ on the space of polynomial sextic differentials and compute their cluster transformations.

Theorem [Neitzke-S. '24]

The image of the harmonic map construction is characterized by the property that $\mathcal{X}_\gamma > 0$.

Thank you!