A classification of modular functors from generalized skein theory

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Outline

- Modular functors are collections of (projective) representations of mapping class groups of surfaces, compatible with cutting and gluing.
- They are closely related to topological field theories (TFT)
- A folklore theorem states that 2d TFT are classified by Frobenius algebras.
- The main goal of the talk is to explain that modular functors are classified by "categorified" Frobenius algebras, satisfying a condition formulated using skein theory.

2d TFT

Definition

Let Bord_2 *be the symmetric monoidal category whose:*

- *objects are disjoint unions of circles*
- *morphisms are equivalence classes of 2-dimensional oriented bordisms*
- *symmetric monoidal structure is given by disjoint union.*

Definition (Atiyah)

An (oriented, non extended) 2d TFT is a symmetric monoidal functor

$$Z : \mathrm{Bord}_2 \longrightarrow \mathrm{Vect}_{\mathbb{K}} .$$

2d TFT

- Evaluation on closed surfaces, seen as bordisms from the empty 1-manifold to itself, gives an element of the base field $\mathbb{K}$.
- Hence a 2d TFT provides numerical invariants of closed surfaces, which can be computed by slicing the surface at hand into elementary pieces.
- Evaluating on some elementary pieces, one gets in particular:
  - a vector space A, the image of S^1
  - a linear map $m : A \otimes A \to A$ from the pair of pants
  - a linear map $1 : \mathbb{K} \to A$ from the "cap"
  - a linear map $\mathrm{tr} : A \to \mathbb{K}$ from the "cup".

Theorem (Folklore)

$(A, m, 1, \mathrm{tr})$ *is a Frobenius algebra, i.e.* $(A, m, 1)$ *is a commutative associative algebra, and* $(a, b) \mapsto \mathrm{tr}(ab)$ *is a non-degenerate invariant pairing. This induces an equivalence between 2d TFTs and Frobenius algebras.*

Why care ?

- Producing invariants of topological surfaces is not so interesting *per se*.
- What makes this interesting is that Frobenius algebras arise naturally in areas which have nothing to do à priori with topology.
- So really this is a way of studying Frobenius algebras graphically.
- A basic example is "gauge theory with a finite group G": the center $Z(\mathbb{K}[G])$ of the group algebra of G is a Frobenius algebra. The corresponding invariant for a closed surface S counts the equivalence classes of representations $\pi_1(S) \to G$.
- In turn this lead to a bunch of useful combinatorial formulas in the representation theory of G.

$2+\epsilon$ **TFT**

- Going one level up, one can consider the (2,1)-category $\widetilde{\text{Bord}}_2$ whose 1-morphisms are 2-dimensional bordisms, and 2-morphisms are isotopy classes of boundary preserving diffeomorphisms.
- A $2+\epsilon$ TFT is a symmetric monoidal functor from $\widetilde{\text{Bord}}_2$ to some symmetric monoidal (2,1) category $\mathcal{S}$, which for the purpose of this talk will be some category of $\mathbb{K}$-linear categories.
- This behaves like the 1 and 2 dimensional part of a (once extended) 3d TFT, hence the name.
- Evaluation on a closed surface S yields a vector space which carries an action of the mapping class group $\text{Map}(S)$ of S.

Modular functors

- This is all well and good, but very often in practice one have to deal with a so-called anomaly: composition of diffeos is preserved only up to twist by a certain cocycle.
- In particular one obtains instead *projective* representations of mapping class groups.
- This leads to the notion of modular functor.
- Perhaps surprisingly there wasn't really a formal definition of a modular functor in the litterature. One of the thing we did with Lukas Woike was to provide one, roughly speaking in terms of representations of central extensions of $\widetilde{\mathrm{Bord}_2}$.

Categorified Frobenius algebra

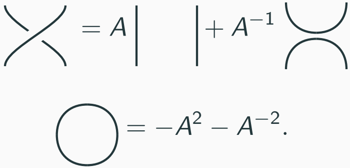
- Just like before, evaluation on the circle and on elementary morphisms gives a category $\mathcal{A}$ and linear functors $m : \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$, $1 : \mathrm{Vect}_{\mathbb{K}} \to \mathcal{A}$ and $\mathrm{tr} : \mathcal{A} \to \mathrm{Vect}_{\mathbb{K}}$.
- This turns $\mathcal{A}$ into a categorified Frobenius algebra (technically: a cyclic algebra over the framed little disc operad).
- Müller–Woike show those are precisely (dualizable) braided monoidal categories which are Grothendieck–Verdier ribbon (this generalizes, and includes, ordinary ribbon categories).
- But the converse of the folklore theorem does not hold, basically because we also need to account for diffeos.

What goes wrong ?

- Müller–Woike showed that every dualizable GV ribbon category $\mathcal{A}$ induces instead an *ansular* functor: like a modular functor, but for handlebodies (with marked discs on their boundary) rather than surfaces.
- To see what's going on, let S be a torus and let a, b be simple closed curves which generates $\pi_1(S)$. There are (essentially unique) handlebodies H_a, H_b bounding S and such that a (resp. b) is contractible in H_a (resp. H_b).
- Thus, we get vector spaces $Z(H_a)$, $Z(H_b)$ carrying actions of $\mathrm{Map}(H_a)$ and $\mathrm{Map}(H_b)$ respectively.
- It's well-known that the restriction map $\mathrm{Map}(H_a) \to \mathrm{Map}(S)$ is injective, and that (the images of) $\mathrm{Map}(H_a)$ and $\mathrm{Map}(H_b)$ generates $\mathrm{Map}(S)$.
- Thus one needs a way to identify $Z(H_a)$ and $Z(H_b)$ in such a way that this descends to an action of $\mathrm{Map}(S)$, but this just isn't possible in general.

Inspiration from skein theory

- Let M be a 3-manifold and $A \in \mathbb{C}$. The (Kauffman) skein module $\mathrm{Sk}(M)$ is the vector space made of linear combinations of isotopy classes of links in M modulo the skein relations



- When A is a root of unity this has a distinguished f.d. quotient $\overline{\mathrm{Sk}}(M)$.
- If S is a surface, $\overline{\mathrm{Sk}}(S) := \overline{\mathrm{Sk}}(S \times I)$ is an algebra by stacking along the interval, and if $S \subset \partial M$, then $\overline{\mathrm{Sk}}(M)$ is an $\overline{\mathrm{Sk}}(S)$-module.
- $\mathrm{Map}(M)$ acts naturally on $\overline{\mathrm{Sk}}(M)$.

Inspiration from skein theory

Theorem (Massbaum–Roberts)

Let S be a (closed, say) surface and let H, H' be two handlebodies bounding S. Then $\overline{\mathsf{Sk}}(H)$ and $\overline{\mathsf{Sk}}(H')$ are isomorphic not just as vector spaces, but as $\overline{\mathsf{Sk}}(S)$-modules. One get this way an action on $\overline{\mathsf{Sk}}(H)$ of a central extension of $\mathrm{Map}(S)$ by the group $\mathrm{Aut}_{\overline{\mathsf{Sk}}(S)}(\overline{\mathsf{Sk}}(M)) \simeq \mathbb{C}^\times$

- One can do something similar for arbitrary surfaces, and this yield a modular functor.
- In that very particular case this is in fact the 2d part of a 3d TFT, namely Witten–Reshetikhin–Turaev theory for quantum SL_2 at a root of unity.

Factorization homology

- Since $\mathcal{A}$ is in particular a (balanced) braided monoidal category, one can apply the formalism of factorization homology: this is a fairly abstract machine that produces for any surface S a category $\int_S \mathcal{A}$, which is part of a twice categorified 2d TFT.
- If $\mathcal{A}$ is semi-simple, essentially coincides with the skein category, having objects configurations of points on S labelled by objects of $\mathcal{A}$, and morphisms linear combinations of graphs embedded in $S \times I$, with vertices labelled by morphisms in $\mathcal{A}$, modulo certain local relations generalizing the Kauffman skein relations.
- There is a distinguished object $\mathcal{O}_S \in \int_S \mathcal{A}$ (the empty configuration).

The main result

Proposition (B-Woike)

Let S be a surface with $n \geq 0$ circle boundary components, and H a handlebody with n marked discs bounding S. This data canonically induces a functor

$$\Phi_H : \int_S \mathcal{A} \longrightarrow \mathcal{A}^{\boxtimes n}$$

such that $\Phi_H(\mathcal{O}_S) = Z(H)$, the value on H of the ansular functor associated with $\mathcal{A}$.

Theorem (B-Woike)

The ansular functor associated with $\mathcal{A}$ extends to a modular functor iff for any surface S and any two handlebodies H, H' bounding S, the functors Φ_H and $\Phi_{H'}$ are isomorphic[a]. We get this way a representation of a central extension of $\mathrm{Map}(S)$ by the center of $\mathrm{Aut}(\Phi_H)$.

[a] as $\mathcal{A}^{\boxtimes n}$-module functors

A few comments

- In the very particular situation of Masbaum–Roberts, $\int_S \mathcal{A}$ turns out to be equivalent to $\overline{\mathsf{Sk}}(S)$-mod, and $\Phi_H = - \otimes_{\overline{\mathsf{Sk}}(S)} \overline{\mathsf{Sk}}(H)$, so we indeed recover their construction in that case.
- A fairly standard argument shows that, in fact, it is enough to check the condition of the theorem in a single case, namely for S a punctured torus, and H, H' any two complementary handlebodies bounding S.
- Crucially, this is indeed a condition, not extra structure, so that if such an extension exists it is unique.
- We also show that "cofactorizability", a much easier condition which can be checked in genus 0, is sufficient.
- Most known examples of modular functors come from modular categories, i.e. braided tensor categories which are ribbon, finite, and non-degenerate (Bakalov–Kirilov, Lyubachenko). Those are known to be cofactorizable so we recover those examples. The uniqueness part, however, is new even in these cases.

Future directions

- The examples coming from modular categories all arise as the restriction of a (at least partially defined) 3d TFT. Since we're asking for less it is natural to expect much more examples, and our result shows that indeed those are pretty special, but I don't know that many examples not of this form.
- In particular, a natural question is whether there exists an example coming from a ribbon category which is finite but not modular (it is known, e.g. that in the finite case a cofactorizable ribbon category is automatically modular)
- Using my previous work with Ben-Zvi–Jordan, the categories $\int_S \mathcal{A}$ and the functors Φ_H can be computed fairly explicitly. In the modular case this should recover the explicit formulas of Lyubachenko.
- I do know examples of modular functors coming from ribbon categories which are not finite, but those are again pretty special, so it would be nice to find other examples.

Thanks for your attention !