TQFT Seminar – Lisbon

Saturday, 6 April 2024 14:17

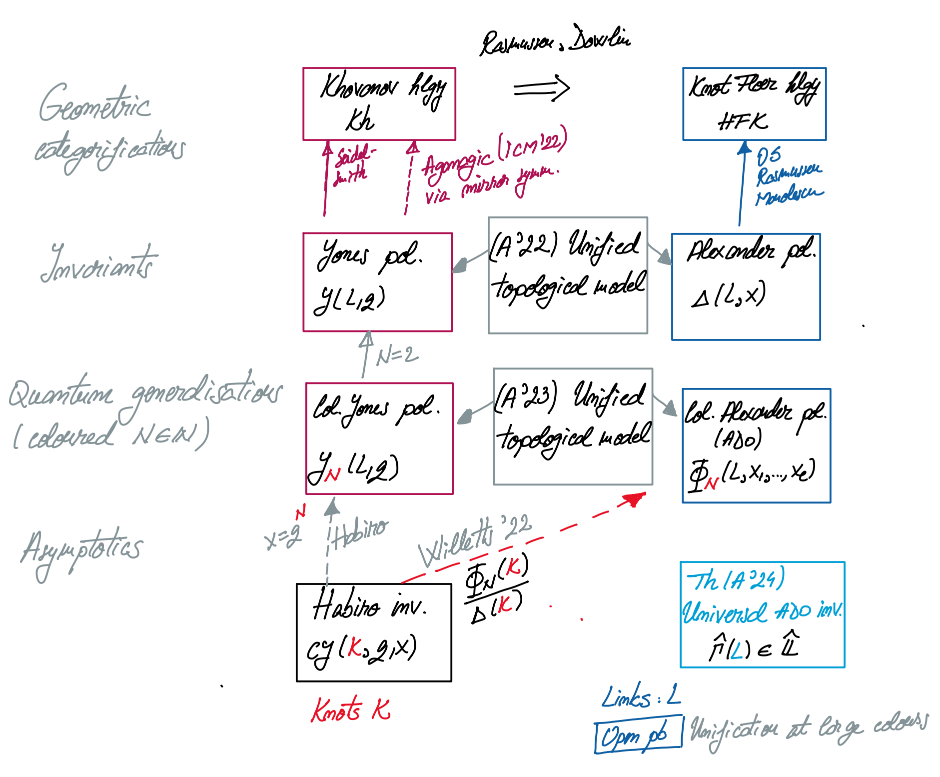
A universal coloured Alexander invariant from configurations on ovals in the disc

- **Outline**
  - (I) Motivation
  - (II) Topological tools
  - (III) Unification of ADO invariants at bounded level
  - (IV) Geometric universal ADO invariant

(I) Topological perspectives on quantum invariants

- **Goal**: Understand quantum invariants from a topological perspective (graded intersections in configuration spaces)

Topological model: graded intersection of two Lagrangians in a configuration space



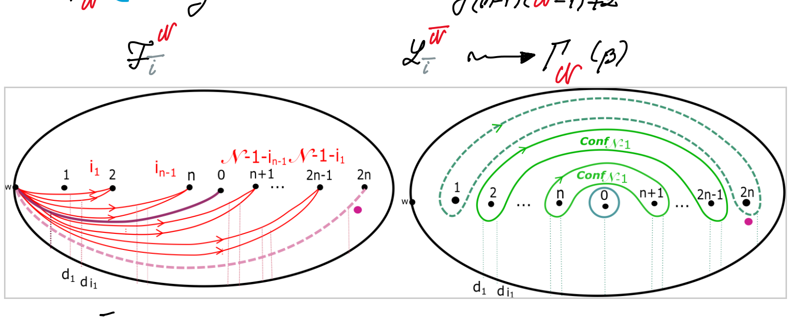
- **Th** (A'24 Universal ADO invariant)
  There exists a link invariant $\hat{\Gamma}(L)\in\hat{\mathbb{L}}$ explicitely given by a limit of graded intersections in configuration spaces s.t.:
  $$\Phi_{\mathcal{N}}(L)=\hat{\Gamma}(L)/_{\psi_{\mathcal{N}}}$$ (specialisation of coeff.)

- **Idea**: The invariant $\hat{\Gamma}(L)$ will be given by a limit of link invariants $\overline{\Gamma}_{\mathcal{N}}(L)$, for $\mathcal{N}\in\mathbb{N}$
  $$\hat{\Gamma}(L)=\lim_{\leftarrow \mathcal{N}}\overline{\Gamma}_{\mathcal{N}}(L)$$
  given by Lagrangian intersections in a fixed config. sp.

- **Context**: $L=\ell$-comp. link $\rightsquigarrow L=\hat{\beta}_n$, $n\in\mathbb{N}$, $\beta_n\in B_n$

$\Gamma_{\mathcal{N}}\leftarrow$ graded intersection in $Conf_{(n-1)(\mathcal{N}-1)+2}(\mathbb{D})$

$\mathcal{F}_{\bar{i}}^{\mathcal{N}}$ $\qquad$ $\mathcal{L}_{\bar{i}}^{\mathcal{N}}\rightsquigarrow\Gamma_{\mathcal{N}}(\beta)$



$\bar{i}=(i_1,\dots,i_{n-1})\in\{\bar{0},\dots,\overline{\mathcal{N}-1}\}$

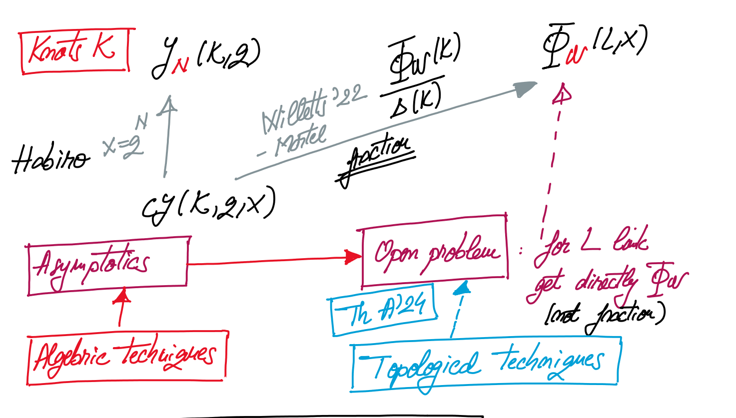
Representation theoretic origin $L=\ell$-comp. link

$Rep(U_q(sl(2)))$ — semisimple / non-semisimple. Fix level $\mathcal{N}\in\mathbb{N}$

(1) q generic
Colour $\overline{N}=(N_1,\dots,N_\ell)\in\mathbb{N}^\ell$
Reshetikhin–Turaev ↓
$$J_{\overline{N}}(L,q)\in\mathbb{Z}[q^{\pm1}]$$
Coloured Jones polyn. (1 var.)

(2) $q=\xi_{\mathcal{N}}=e^{2\pi i/2\mathcal{N}}$
Akutsu–Deguchi–Ohtsuki ('91) (ADO) ↓
$\Lambda_\ell=\mathbb{Q}(\xi_{\mathcal{N}})[x_1^{\pm1},\dots,x_\ell^{\pm1}]$ $(x_1^{2\mathcal{N}-1},\dots,x_\ell^{2\mathcal{N}-1})$
$$\Phi_{\mathcal{N}}(L,x_1,\dots,x_\ell)\in\Lambda_\ell$$
Coloured Alexander polyn. (ADO) (ℓ var.)

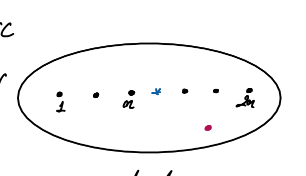
- **Knots vs Links**



- **Why quantum invariants for links?**
  Building blocks for 3-mfd. invariants ← Topological meaning is an important open problem

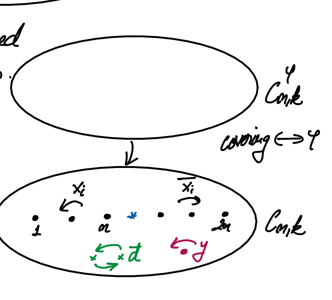
(II) Topological tools $L=\hat{\beta}_n$

$\mathbb{D}=\mathbb{D}_{2n+2}$-punctured disc
- $2n$ horizontal punctures
- 1 q-puncture
- 1 s-puncture



For $k\in\mathbb{N}$: $C_{n,k}:=Conf_k(\mathbb{D})$ unordered conf. sp.

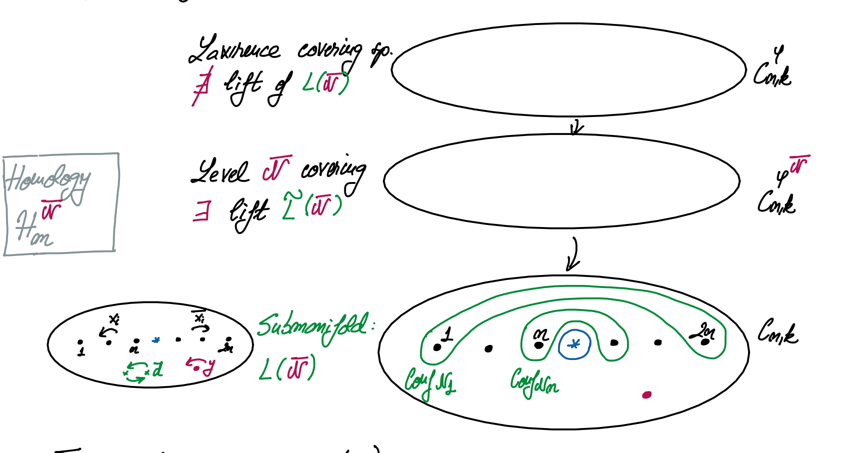
- **Def**: (Local system)
  $$\pi_1(C_{n,k})\xrightarrow{\varphi}\mathbb{Z}^n\oplus\mathbb{Z}^n\oplus\mathbb{Z}\oplus\mathbb{Z}$$
  $\langle x_i\rangle$ $\langle\bar{x}_i\rangle$ $\langle y\rangle$ $\langle d\rangle$



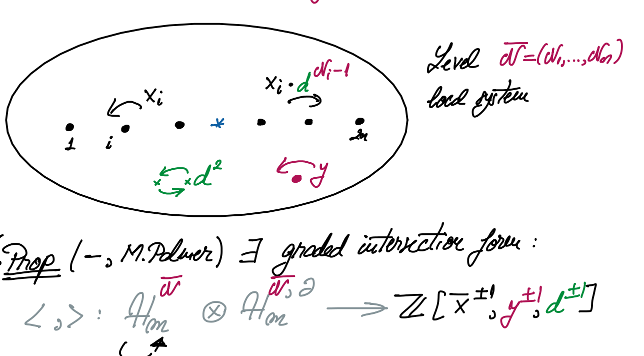
- **Idea**:
  Intersections in covering sp. associated to φ ⟷ Intersections in the • base sp • graded by φ

- Coming from repr. thy. we want to work with submanifolds whose geometric supp. is given by config. sp. on ovals, of prescribed multiplicity

- Fix $\overline{\mathcal{N}}=(\mathcal{N}_1,\dots,\mathcal{N}_n)$ partition of $n-1$



- **Def**: (Level $\overline{\mathcal{N}}$ local system)
  $$\pi_1(C_{n,k})\xrightarrow{\varphi}\mathbb{Z}^n\oplus\mathbb{Z}^n\oplus\mathbb{Z}\oplus\mathbb{Z}$$
  $\langle x_i\rangle$ $\langle\bar{x}_i\rangle$ $\langle y\rangle$ $\langle d\rangle$
  $\varphi^{\overline{\mathcal{N}}}$ ↘ ↓ $f^{\overline{\mathcal{N}}}$ ($\bar{x}_i\mapsto x_i\cdot d^{\mathcal{N}_i-1}$, $d\mapsto d^2$)
  $$\mathbb{Z}^n\oplus\mathbb{Z}\oplus\mathbb{Z}$$
  $\langle x_i\rangle$ $\langle y\rangle$ $\langle d\rangle$



Level $\overline{\mathcal{N}}=(\mathcal{N}_1,\dots,\mathcal{N}_n)$ local system

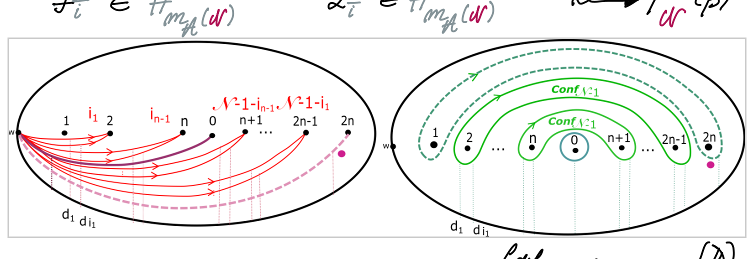
- **Prop** (–, M.Palmer) ∃ graded intersection form:
  $$\langle\,,\rangle: H_n^{\overline{\mathcal{N}}}\otimes H_n^{\overline{\mathcal{N}},\partial}\longrightarrow\mathbb{Z}[\bar{x}^{\pm1},y^{\pm1},d^{\pm1}]$$
  B_{2n+2} acts.

(III) Topological model for the $\mathcal{N}^{th}$ ADO invariant

Fix $\mathcal{N}$ – level for the ADO pol. $\Phi_{\mathcal{N}}(L)$

- **Parameters**
  - # pts in the config. sp. $m_A(\mathcal{N})=2+(n-1)(\mathcal{N}-1)$
  - Multi-level: $\overline{\mathcal{N}}:=(1,\mathcal{N}-1,\dots,\mathcal{N}-1)$

- **Def**: (Homology classes) $\bar{i}=(i_1,\dots,i_{n-1})\in\{\bar{0},\dots,\overline{\mathcal{N}-1}\}$
  $\mathcal{F}_{\bar{i}}^{\mathcal{N}}\in H_{m_A(\mathcal{N})}^{\overline{\mathcal{N}}}$ $\qquad$ $\mathcal{L}_{\bar{i}}^{\mathcal{N}}\in H_{m_A(\mathcal{N})}^{\overline{\mathcal{N}}}$ $\rightsquigarrow\Gamma_{\mathcal{N}}(\beta)$



$Conf_{(n-1)(\mathcal{N}-1)+2}(\mathbb{D})$

⟶ Lagrangian intersection
$$\langle(\beta_n\cup\mathbb{1})\mathcal{F}_{\bar{i}}^{\mathcal{N}},\mathcal{L}_{\bar{i}}^{\mathcal{N}}\rangle\in\mathbb{Z}[x_1^{\pm1},\dots,x_\ell^{\pm1},y^{\pm1},d^{\pm1}]$$

- **Def**: (Ring of coeff.)
  $\mathbb{L}:=\mathbb{Z}[u_1^{\pm1},\dots,u_\ell^{\pm1},x_1^{\pm1},\dots,x_\ell^{\pm1},y^{\pm1},d^{\pm1}]$
  $\psi_{\mathcal{N}}\downarrow$: $u_i\mapsto x_i^{1-\mathcal{N}}$, $y\mapsto\frac{x_1-x_1^{-1}}{x_1^{\mathcal{N}}-x_1^{-\mathcal{N}}}$, $d\mapsto\xi_{\mathcal{N}}=e^{2\pi i/2\mathcal{N}}$
  $\Lambda_{\mathcal{N}}$

- **Def**: (Graded intersection)
  $$\Gamma_{\mathcal{N}}(\beta_n):=f(\bar{u})\cdot\sum_{\bar{i}=\bar{0}}^{\overline{\mathcal{N}-1}}\langle(\beta_n\cup\mathbb{1})\mathcal{F}_{\bar{i}}^{\mathcal{N}},\mathcal{L}_{\bar{i}}^{\mathcal{N}}\rangle\in\mathbb{L}$$

- **Th** (A'24 Topological model for the $\mathcal{N}^{th}$ ADO inv.)
  Let L – link and $\beta_n\in B_n$ s.t. $L=\hat{\beta}_n$. Then:
  $$\Phi_{\mathcal{N}}(L)=\Gamma_{\mathcal{N}}(\beta_n)/_{\psi_{\mathcal{N}}}$$

∴ We can read-off the $\mathcal{N}^{th}$ ADO invariant from a fixed set of intersection points between Lagrangian submanifolds in $Conf_{(n-1)(\mathcal{N}-1)+2}(\mathbb{D})$

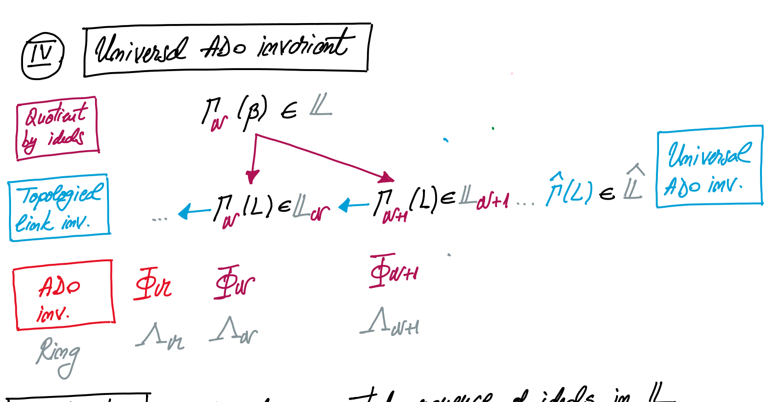
- **Q**: Asymptotic behaviour?
  Phenomena when changing the colour $\mathcal{N}\to\mathcal{N}+1\to\dots$

- **Th** (A'24 Unification of all ADO invariants up to level $\mathcal{N}$)
  Let $\mathcal{N}\in\mathbb{N}$ be a fixed level. Then:
  $$\Phi_{\mathcal{M}}(L)=\Gamma_{\mathcal{N}}(\beta_n)/_{\psi_{\mathcal{M}}},\ \forall\,\mathcal{M}\leq\mathcal{N}.$$

∴ We can read-off all ADO invariants at levels less than $\mathcal{N}$ from a fixed set of intersection points in $Conf_{(n-1)(\mathcal{N}-1)+2}(\mathbb{D})$

- **Proof**: Topological, and uses in an essential manner the geometry of the classes $\mathcal{F}^{\mathcal{N}}$
  We used the classes $\mathcal{L}^{\mathcal{N}}$ to make sure we can define the intersection
  For that, we needed a well-chosen local system on the configuration sp. to ensure we can lift submanifolds supported by config. on ovals

(IV) Universal ADO invariant

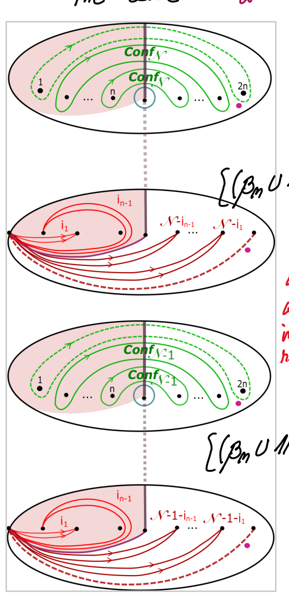


Construction
- Consider a nested sequence of ideals in $\mathbb{L}$
  $I_{\mathcal{N}}\supseteq I_{\mathcal{N}+1}\supseteq\dots$ → $\mathbb{L}_{\mathcal{N}}:=\mathbb{L}/I_{\mathcal{N}}$
- Well-defined limit: $\hat{\mathbb{L}}:=\lim_{\leftarrow\mathcal{N}}\mathbb{L}_{\mathcal{N}}$

- **Th** (A. Link invariants at level $\mathcal{N}$)
  $\overline{\Gamma}_{\mathcal{N}}(L)\in\mathbb{L}_{\mathcal{N}}$ is an invariant of L which recovers all ADO invariants $\Phi_{\mathcal{M}}(L)$, $\forall\,\mathcal{M}\leq\mathcal{N}$.
  (given by a fixed set of intersection pts. in $Conf_{(n-1)(\mathcal{N}-1)+2}(\mathbb{D})$)

- **Th** (A'24 Universal ADO invariant)
  There exists a well-defined limit
  $$\hat{\Gamma}(L):=\lim_{\leftarrow}\overline{\Gamma}_{\mathcal{N}}(L)\in\hat{\mathbb{L}}$$
  which is a link invariant recovering all ADO invariants:
  $$\Phi_{\mathcal{M}}(L)=\hat{\Gamma}(L)/_{\psi_{\mathcal{M}}^{u}}$$

- **Proof**: Key points:
  (1) $\Gamma_{\mathcal{N}}$ contains all ADO $\Phi_{\mathcal{M}}$ for $\mathcal{M}\leq\mathcal{N}$
  (2) Good topological behaviour when changing the level: $\Gamma_{\mathcal{N}}\to\Gamma_{\mathcal{N}+1}$



$\{(\beta_n\cup\mathbb{1})\mathcal{F}_{\bar{i}}^{\mathcal{N}+1}\cap\mathcal{L}_{\bar{i}}^{\mathcal{N}+1}\}_{\bar{i}\in\bar{0},\overline{\mathcal{N}}}\to\overline{\Gamma}_{\mathcal{N}+1}(L)$

add arcs in the r.h.s of $\mathbb{D}$ | add one particle on each circle

$\{(\beta_n\cup\mathbb{1})\mathcal{F}_{\bar{i}}^{\mathcal{N}}\cap\mathcal{L}_{\bar{i}}^{\mathcal{N}}\}_{\bar{i}\in\bar{0},\overline{\mathcal{N}-1}}\to\overline{\Gamma}_{\mathcal{N}}(L)$

Correspondence quantum to homological … and back again

New questions – – ? – – Universal ADO inv $\hat{\Gamma}$

Open ↑ $\mathcal{N}\to\infty$ ↑ (3)

ADO inv. $\{\Phi_{\mathcal{N}}\}_{\mathcal{N}}$, $\mathcal{N}-1$, ⋮ ← (1) (2) — $\{\Gamma_{\mathcal{N}}\}_{\mathcal{N}}$

Repr. thy — Topological model