Local constructions of exotic Lagrangian tori.

Lisbon Geometry Seminar – 2.4.24

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Based on:

- Local exotic tori (arXiv: 2310.11359)

  dim ≥ 6

- Semi-local exotic tori in dim. four j.w/ Johannes HAUBER, Joel SCHMITZ (arXiv: 2403.00408)

  dim $= 4$

Overview: I. Question & Motivation
II. "state of the art" & Results
III. Locality
IV. Dimension four
V. Locality: idea of proof.

I. Question & Motivation

(X^{2n}, ω) symplectic manifold

Question: (*) Classify compact Lagrangian submanifolds up to

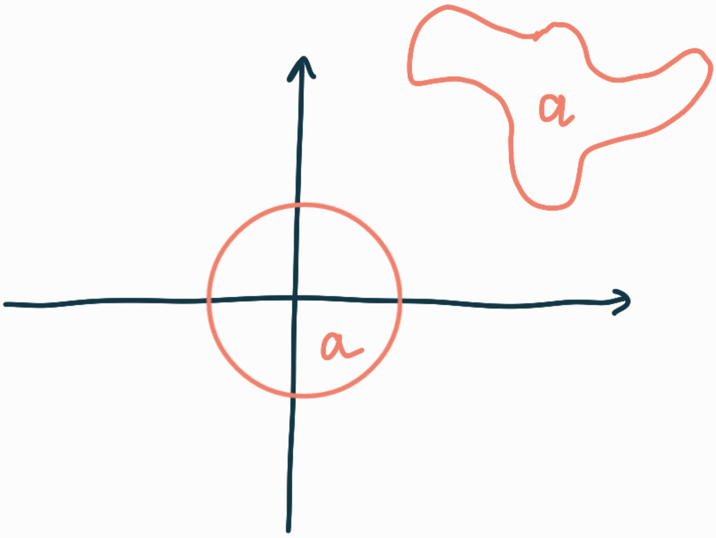
•) $\mathrm{Symp}(X,\omega) = \{\phi \in \mathrm{Diff}(X) \mid \phi^*\omega = \omega\}$

or

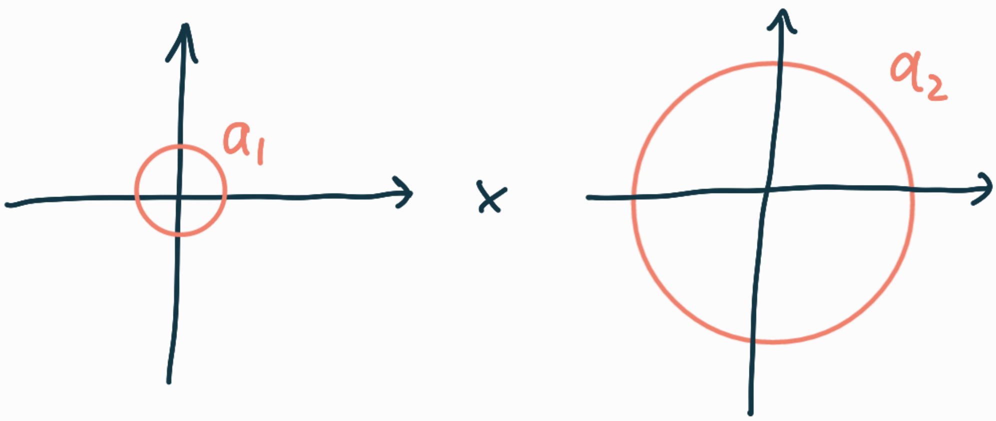
•) $\mathrm{Ham}(X,\omega) = \{$flows of time-dep. Hamiltonian v. fields$\}$

Example: $(\mathbb{R}^2, \omega_0 = dx \wedge dy)$

compact Lagrangian submanifolds : closed embedded curves.



Example': $(\mathbb{R}^4, \omega_0)$



$T(a_1, a_2)$: product torus

$a_1, a_2 > 0$

Are there other Lagrangian tori?
(not Symp/Ham-equiv. to product tori)

Chekanov '96 : Yes.

$T_{Ch}(a) \subset \mathbb{R}^4 \quad a > 0$
(scaling)

Chekanov torus.

(see also Eliashberg - Polterovich '95)

<u>Note</u>: 1) "Classical" / "soft" invariants

$\alpha_L, \mu_L : \pi_2(X, L) \longrightarrow \mathbb{R}, \mathbb{Z}$

area / Maslov class

do NOT distinguish

$T_{Ch}(a)$ from $T(a,a)$.

2) This is an instance of

"symplectic rigidity":

there is a smooth isotopy of

$\mathbb{R}^4$ mapping $T_{Ch}(a)$ to $T(a,a)$.

Question ⊛ is open for $\mathbb{R}^4$!

(very difficult in general ...)

Instead : 1) Restrict attention to a sub-class of Lagrangian tori

e.g. product tori (Chekanov)

toric fibres (see my last LGS-talk ☺)

2) Try to come up with new exotic tori.

Definition: i) $L, L' \subset (X, \omega)$ area-equivalent if

•) Lagrangian isotopic

•) same classical invts.

ii) $L, L' \subset (X, \omega)$ (symplectically) inequivalent if

$\nexists\, \phi \in Symp(X, \omega)$ s.th. $\phi(L) = L'$.

In general (X, ω), we look for sets $L_1, L_2, \ldots \subset X$ of Lagrangian tori which are pairwise i) area-equivalent

ii) inequivalent.

II. State of the art & Results

dim $= 4$	dim ≥ 6
$\mathbb{R}^4$: 2 (Chekanov '96)	$\mathbb{R}^6$: ∞ (Auroux '15)
$\mathbb{CP}^2$, $S^2 \times S^2$, Del Pezzo : ∞ (Vianna '17)	previous work by: (Chekanov, Chekanov-Schlenk ...)
(previous/other work by: Biran, Entov-Polterovich, Chekanov-Schlenk, Gadbled, Abreu-Gadbled, FOOO, Wu, Oakley-Usher, Shelukhin-Tonkonog-Vianna ...)	

<u>Results</u> :

(dim $= 4$)	(dim ≥ 6)
X compact and admits (almost) toric structure : ∞ (B.-Hauber-Schmitz '24)	X geometrically bounded : ∞ (B. '23)

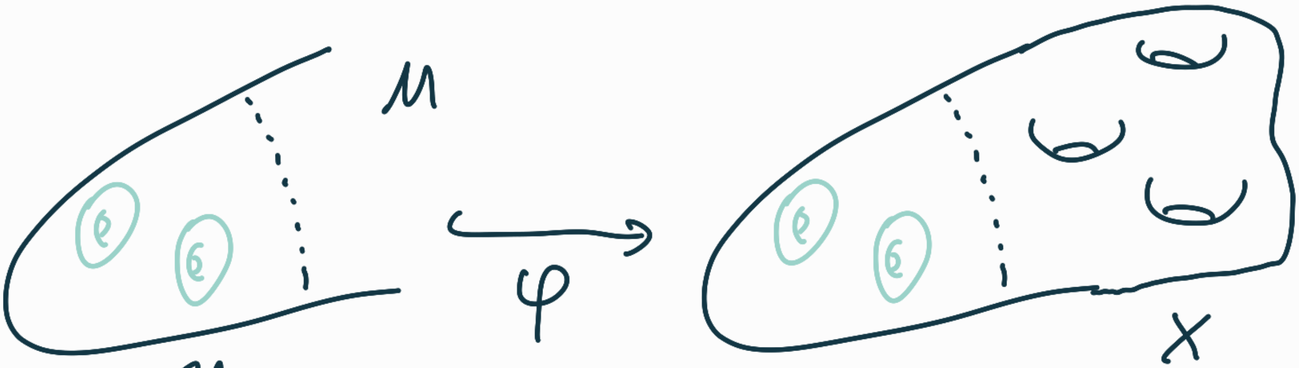
III. Locality

model space : general (X, ω) :

$$(M, \omega_M) \supset \mathcal{U} \xhookrightarrow{\varphi} (X, \omega)$$

$$\mathcal{U} \supset L \not\simeq L' \qquad (X,\omega) \supset \varphi(L) \not\simeq \varphi(L')$$

(in M) (in X)



$\mathcal{U}$

Example : (dim ≥ 6 B. '23)

$$(\mathbb{R}^{2n}, \omega_0) \supset B^{2n}(R) \xhookrightarrow{\varphi} (X, \omega)$$

(model space)

$$\underbrace{B^{2n}(R) \supset T_0, T_1, T_2, \ldots}_{\text{(known by Auroux)}} \qquad (X,\omega) \supset \varphi(T_0), \varphi(T_1), \varphi(T_2), \ldots$$

Darboux !

IV. Dimension four

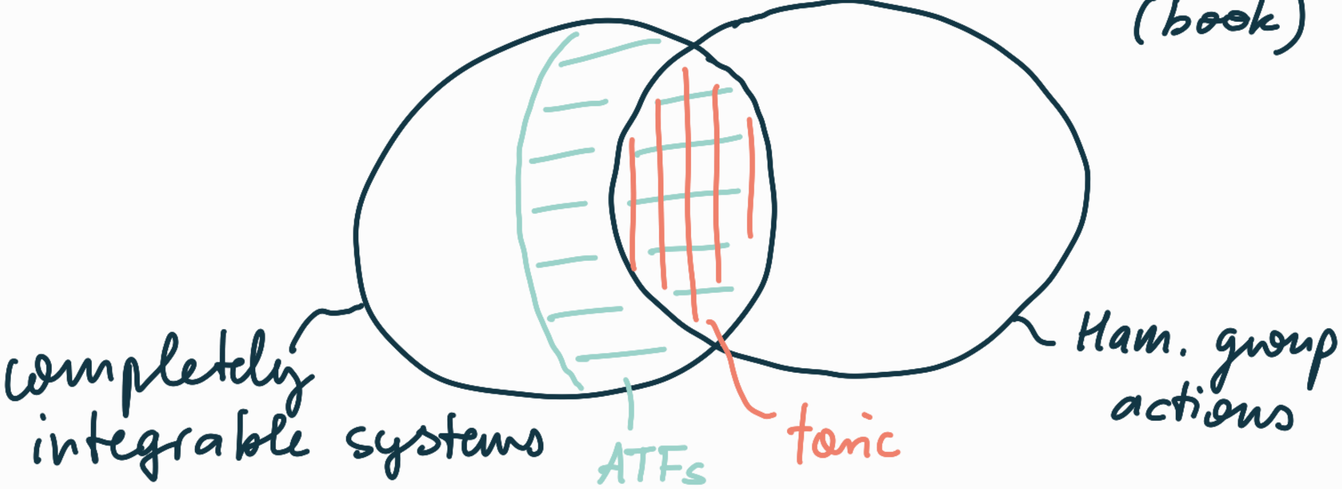
<u>Note</u>: $\mathbb{R}^4$ is not a good choice of model space in dim = 4.
(only product and Chekanov tori)

↪ need more complicated model spaces.

(Drawback: Symplectic embedding is not for free!)

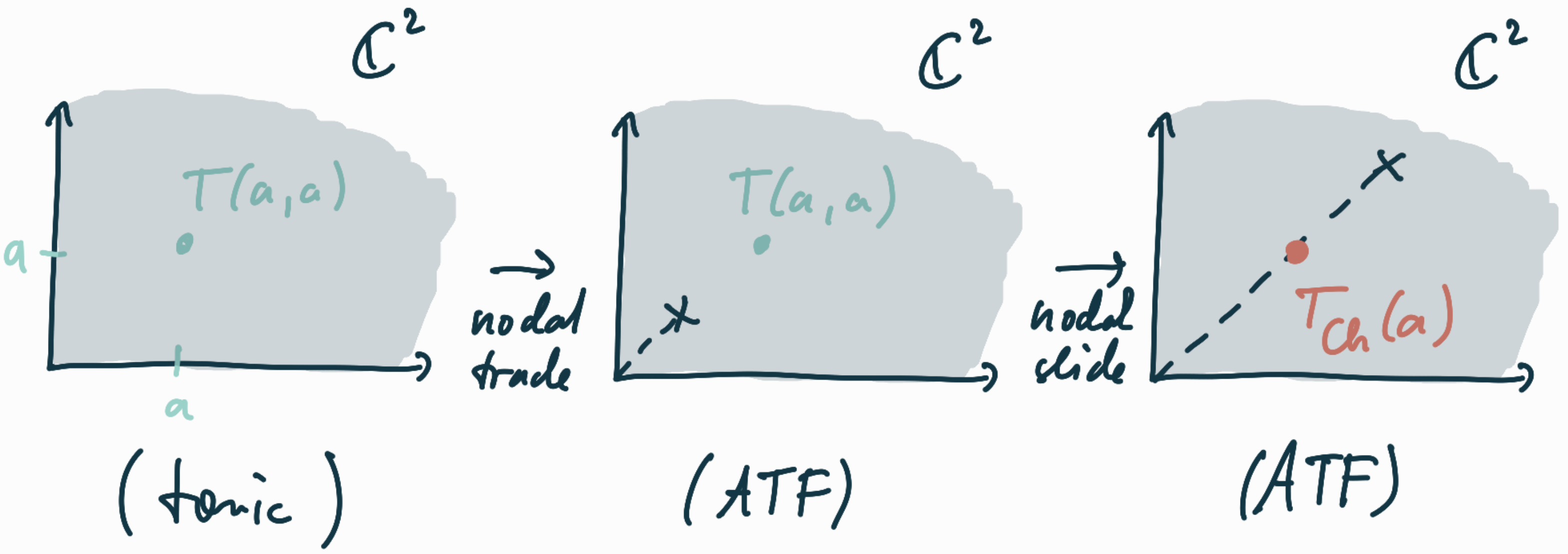
<u>Excursion</u>: Almost toric fibrations (ATFs)

Generalization of toric manifolds in dimension four, due to
Zung, Symington, Vianna, <u>Evans</u>, ...
(book)



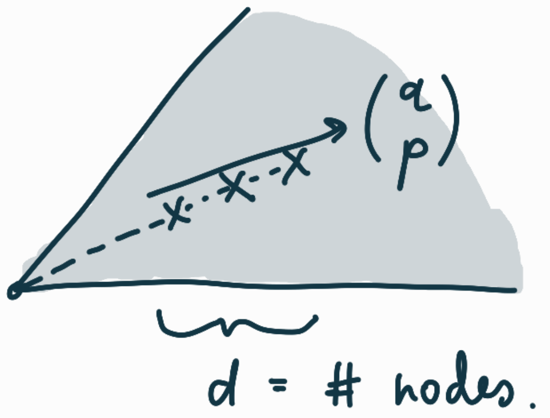
ATFs yield exotic tori as their fibres (Vianna '17).

<u>Example</u>:



ATF base diagram $\rightsquigarrow$ (Sym. '03) unique sympl. manifold

Model spaces : $B_{d,p,q}$ $\qquad d,p,q \in \mathbb{N}$ (q mod p)



obtained from all non-compact ATF base diagrams with one single vertex.

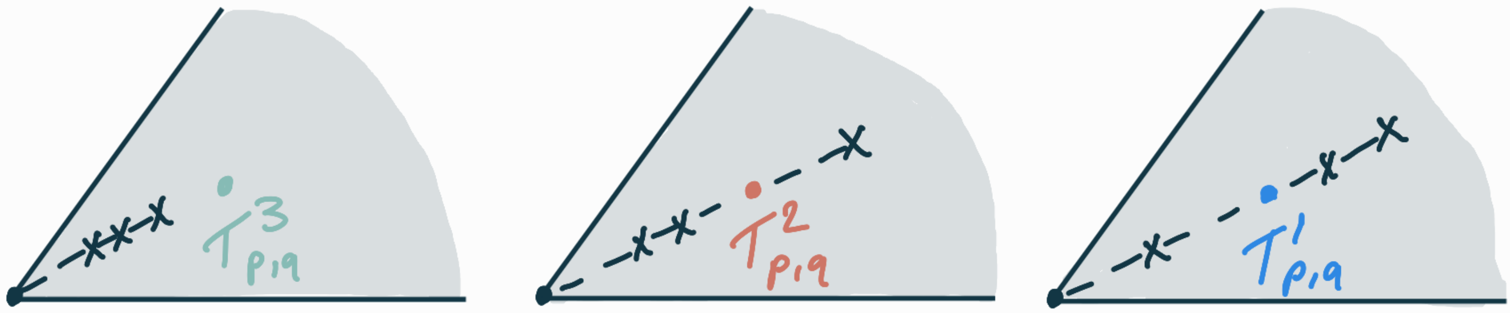
Other points of view :

1) $B_{d,p,q}$'s are Milnor fibres of certain cyclic quotient singularities.

2) $B_{d,p,q}$'s are symplectic fillings of certain lens spaces.

(see J. Evans
*) LTF - book § 7.4
*) KIAS lecture notes)

The Lagrangian tori appear as fibres of these ATFs.



Thm A: (B.-H.-S.)

$$T^0_{p,q}, \dots, T^d_{p,q} \subseteq B_{d,p,q}$$

non-equivalent tori.

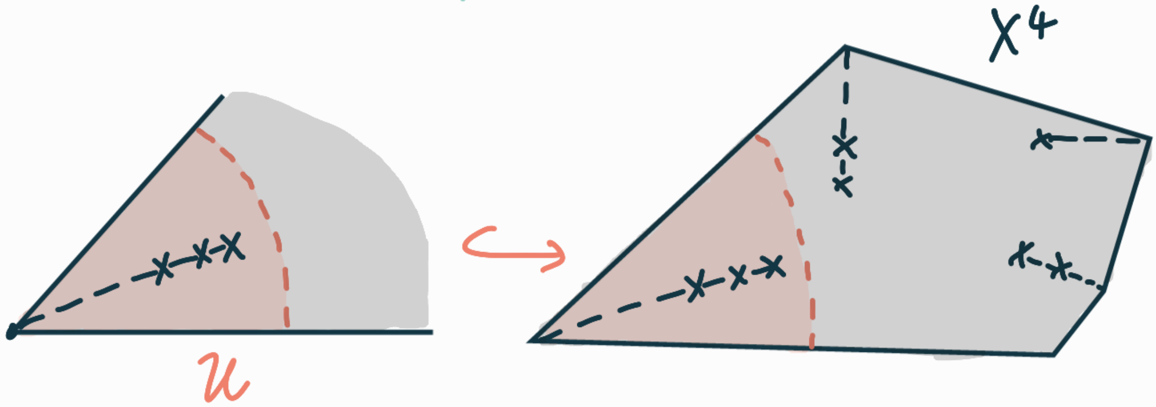
Thm B: (B.-H.-S.)

Let (X^4, ω) be a compact almost toric manifold. Then X^4 contains infinitely many inequivalent tori.

idea: 1) By almost toric mutation, produce ∞-many embeddings

$$B_{d_i, p_i, q_i} \supset \mathcal{U}_i \hookrightarrow (X^4, \omega)$$

with $p_i \to \infty$.



2) Use locality and the fact that there is a symplectic invariant of $\varphi(T^k_{p,q})$ which recovers k, p, q.

<u>Example</u>: $B_{2,1,0} = T^*S^2$

Thm A $\Rightarrow$ 3 different Lagrangian tori

$T^0_{1,0}$ = Chekanov torus

$T^1_{1,0}$ = Clifford torus

$T^2_{1,0}$ = Polterovich torus

(Albers – Frauenfelder '05)

<u>Corollary</u>:

Let (X^4, ω) geom. bounded symplectic manifold containing a Lagrangian sphere. Then X^4 contains 3 inequivalent tori.

<u>proof</u>: Weinstein & locality $\square$

("semi-local": using Weinstein near Lagrangians instead of Darboux.)

II. Locality : ideas of proof

Symplectic invariant : displacement energy.

$A \subset (X, \omega)$ compact

$$e(A) := \left\{ \|\phi\| \;\middle|\; \begin{array}{l} \phi \in Ham(X,\omega) \\ \phi(A) \cap A = \emptyset \end{array} \right\}$$

Hofer norm

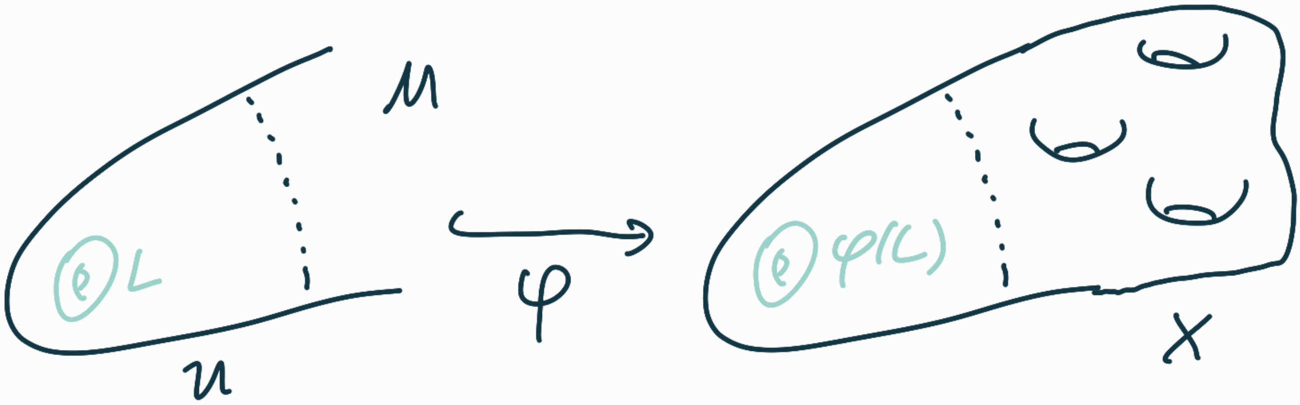
combined with "versal deformations"
(Chekanov '96)

1) Compute $e(\cdot)$ on a neighbourhood of $L \in \mathcal{L}$:= space of Lagrangians

2) Take its "germ at L"

$\rightsquigarrow$ $\boxed{\mathcal{E}_L : H^1(L;\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}}$
Displacement energy germ

Locality of displacement energy germ:



For small enough for $L \subset M$, have

$$\boxed{\mathcal{E}_L = \mathcal{E}_{\varphi(L)}}$$

(!) This is non-trivial:
$\exists\, \mathcal{U} \subset \mathcal{L}$ nbhd of L s.th.
$\forall L' \in \mathcal{U}$, we have:

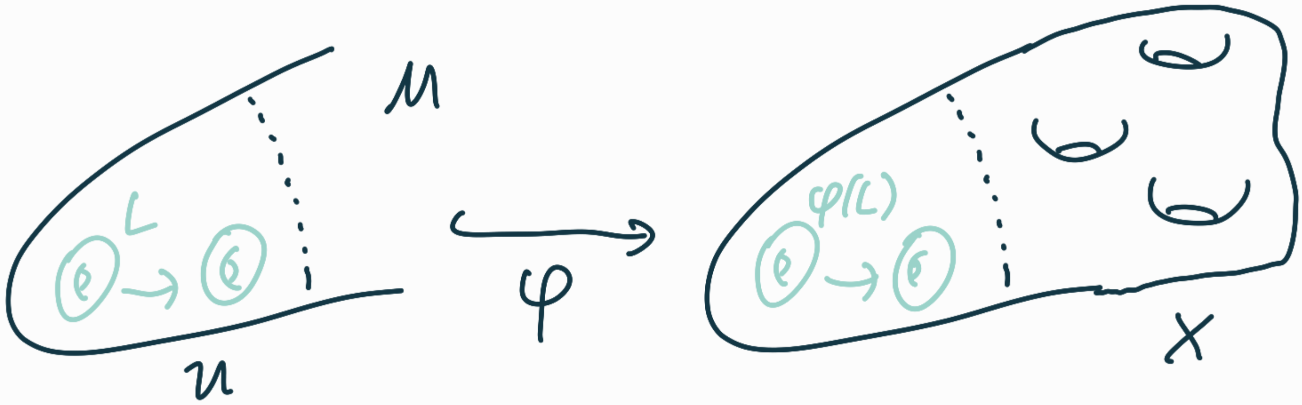
$$e_M(L') = e_X(\varphi(L'))$$

displacement energy in different spaces.

1) $\boxed{e_X(\varphi(L')) \leq e_M(L')}$

explicit displacement ("soft")

If L is small enough:



displacing L in $\mathcal{U}$ $\overset{\text{cut-off}}{\Longrightarrow}$ displacing $\varphi(L)$ in $\varphi(\mathcal{U})$.

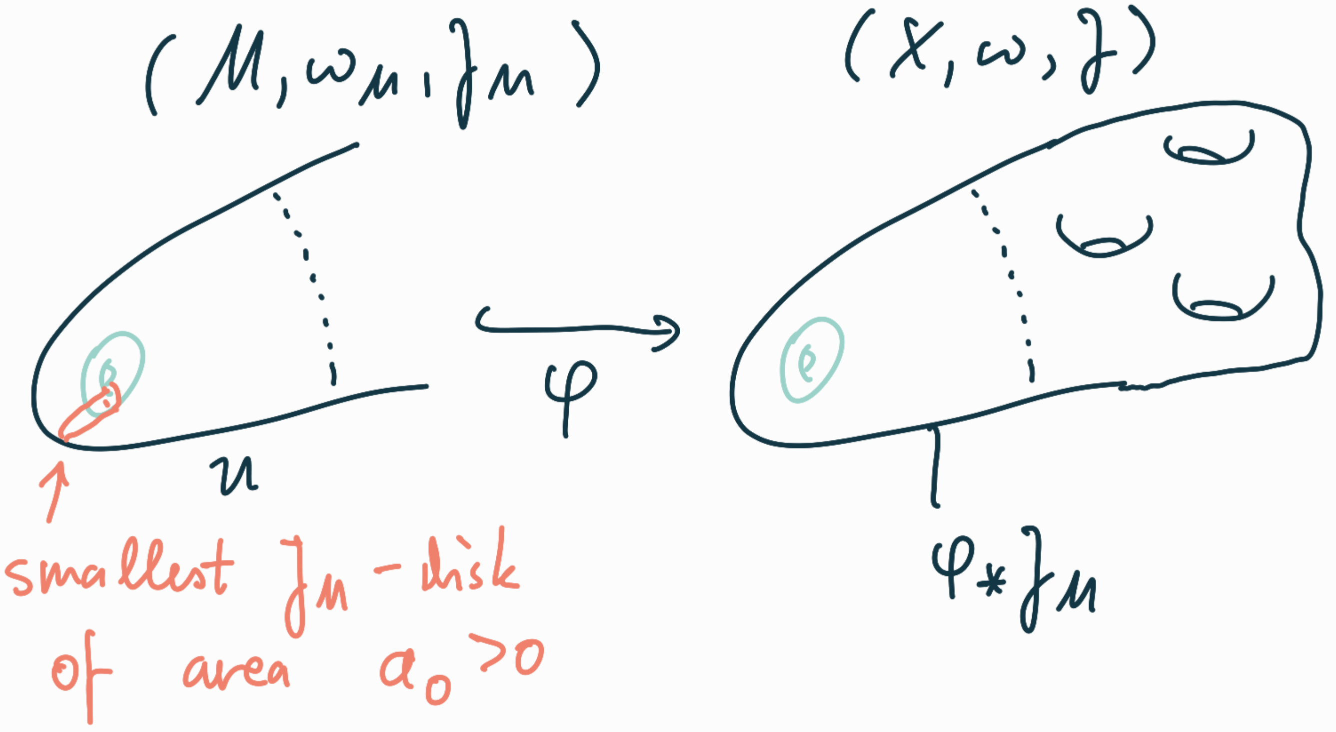
2) $\boxed{e_X(\varphi(L')) \geq e_M(L')}$

<u>Theorem:</u> (Chekanov '98)

$L \subset (X,\omega) \leftarrow$ geom. bounded

L compact

$$e(L) \geq \sup_{J \in \mathcal{J}_{tame}} \left\{ \begin{array}{l} \text{min area of } J\text{-} \\ \text{spheres and } J\text{-disks} \\ \text{w/ bdry on } L \end{array} \right\}$$



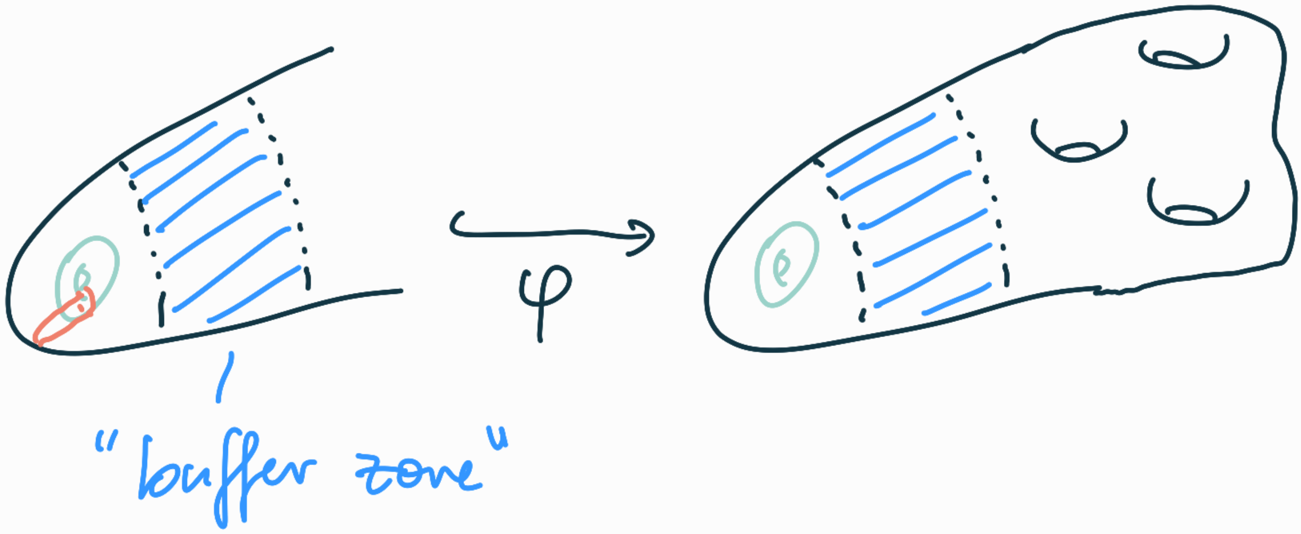
Choose c small enough s.th.

1) $a_0 <$ area of smallest J-sphere in X

(this is the only input from (X, ω) !)

2) $a_0 <$ area of J-disks leaving $\varphi(\mathcal{U})$

To acheive this :



<u>Lemma</u>: If a J-holomorphic curve crosses , it has area $> a_0$.

For $M = \mathbb{R}^{2n}$, this Lemma was proved by Chekanov – Schlenk, for $M = B_{d,p,q}$ in B. – H. – S.


Thank You!