Hamiltonian Actions on Symplectic Manifolds

Leonor Godinho
Instituto Superior Técnico

WSM, 14 Mar, 2024

- **Manifold** M

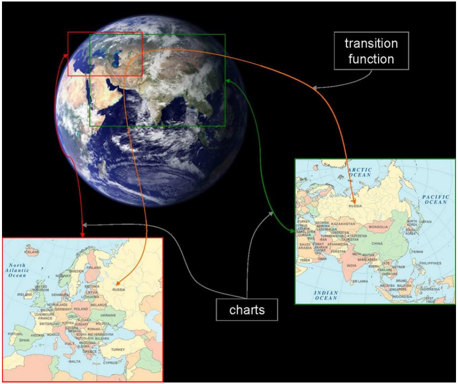


Figure: Jean Gallier (Construction of C^∞ Surfaces from Triangular Meshes Using Parametric Pseudo-Manifolds)

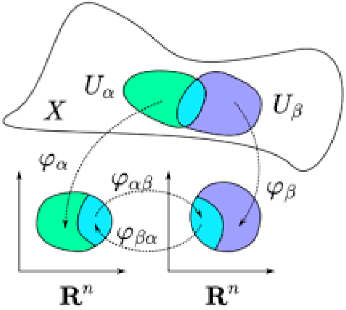


Figure: Wikipedia

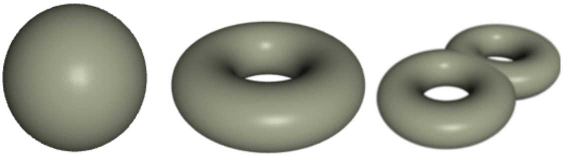


Figure: D. Warne in *On the Effect of Topology on Cellular Automata Rule Spaces*

- **Symplectic** manifold (M, ω) - manifold M +

  closed non-degenerate 2-form ω - **symplectic form**.

- **Symplectic** manifold (M, ω) - manifold M +

  closed non-degenerate 2-form ω - **symplectic form**.

- a 2-form, ω, is a **bilinear, skew-symmetric** function that for each point $p \in M$ and tangent vectores $X_p, Y_p \in T_pM$ associates a real number $\omega_p(X_p, Y_p)$.

$$\omega_p : T_pM \times T_pM \to \mathbb{R}$$

- **Symplectic manifold** (M, ω) - manifold M +

  closed non-degenerate 2-form ω - **symplectic form.**

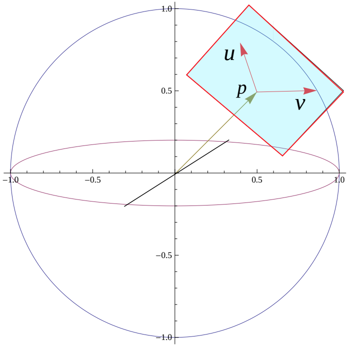
- closed: $d\omega = 0$
- non-degenerate:

$$\forall p \in M, \text{ if } \omega_p(X_p, Y_p) = 0 \text{ for all } Y_p \in T_pM, \text{ then } X_p = 0.$$

$\Leftrightarrow$ Each nonzero $v \in T_pM$ has a symplectic best friend $w \in T_pM$ s.t.

$$\omega_p(v, w) = 1.$$

Example: The sphere $S^2 = \{p \in \mathbb{R}^3 : ||p|| = 1\}$



$$u, v \in T_pS^2 = \{p\}^{\perp} \qquad \omega_p(u, v) := \langle p, u \times v\rangle = \det(p, u, v).$$

Symplectic Manifolds

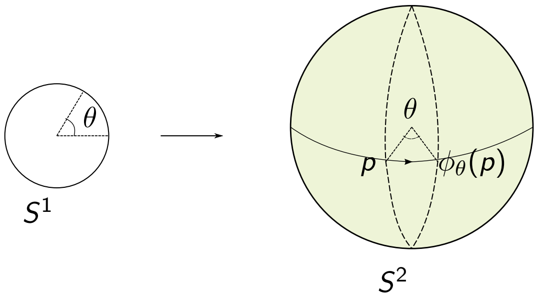
Teorema de Darboux : *Locally* all symplectic forms are identical:

Local model - $(\mathbb{R}^{2n}, \omega_0)$.

$$p = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}$$

$$\omega_0 = \sum_{k=1}^{n} dx_k \wedge dy_k$$

Symplectic action: An example



Each $\theta \in S^1$ can be seen as a transformation

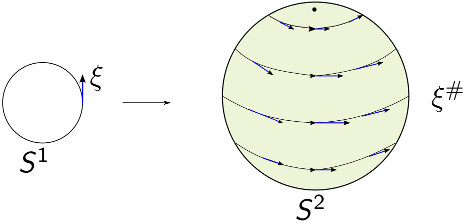
$$\phi_\theta : S^2 \to S^2$$

that preserves the symplectic form.

Hamiltonian action: An example

$S^1 \curvearrowright (M, \omega)$

$\xi^\#$: Vector field associated to the action of S^1.

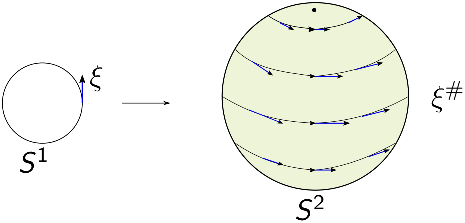


For $p = (x, y, z)$, we have:

$$\xi^\#_p = \frac{\partial \phi_\theta}{\partial \theta}(x, y, z)_{|\theta=0} = \frac{\partial}{\partial \theta}\left(\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}(x, y), z\right)\right)_{|\theta=0}$$
$$= (-y, x, 0)$$

$S^1 \curvearrowright (M, \omega)$

$\xi^\#$: Vector field associated to the action of S^1.



$$\iota_{\xi^\#}\omega = \omega(\xi^\#, \cdot) \to \text{1-form}$$

- **Hamiltonian Action**: $\iota_{\xi^\#}\omega$ is **exact**, i.e. $\exists H \in C^\infty(M)$ s.t.

$$\iota_{\xi^\#}\omega = dH.$$

$H\colon M \to \mathbb{R}$ - **Moment Map**

Hamiltonian action: An example

$$S^1 \curvearrowright (M,\omega), \quad \omega_p(u,v) = det(p,u,v), \;\; u,v \in T_pS^2$$

For $p = (x,y,z)$, we have:

1. $\xi_p^{\#} = (-y, x, 0)$
2. For $v = (a,b,c) \in T_pS^2$,

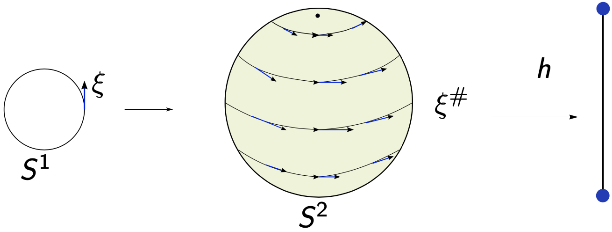
$$\omega_p(\xi_p^{\#}, v) = det(p, \xi_p^{\#}, v) = det\begin{bmatrix} x & -y & a \\ y & x & b \\ z & 0 & c \end{bmatrix}$$
$$= -xza - yzb + (x^2+y^2)z =_* z = dz(x,y,z)$$

since $\langle p, v\rangle = 0$ implies

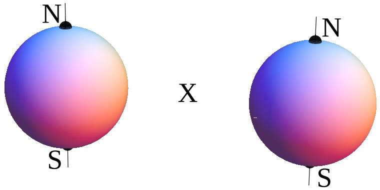
$$xa + yb + cz = 0.$$

Hamiltonian action: An example

$\omega_p(\xi_p^{\#}, \cdot) = dh$ with $h(x, y, z) = z$.



Example: $G = \mathbb{T}^2 = S^1 \times S^1 \curvearrowright (S^2 \times S^2, \omega \oplus \omega)$



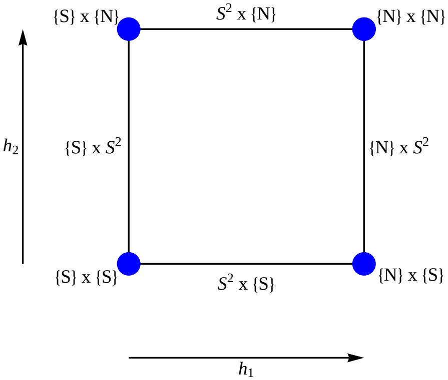
- The 1st circle rotates the 1st sphere, fixing the 2nd one;
- The 2nd circle rotates the 2nd sphere, fixing the 1st one.

4 **Fixed Points**: $S \times S$, $S \times N$, $N \times S$ and $N \times N$.

Two Hamiltonian circle actions

$$H = (h_1, h_2) : S^2 \times S^2 \to \mathbb{R}^2 \to \textbf{Moment Map}$$

Image of the Moment Map $H = (h_1, h_2) : S^2 \times S^2 \to \mathbb{R}^2$



Example: $G = \mathbb{T}^3 = S^1 \times S^1 \times S^1 \curvearrowright (S^2 \times S^2 \times S^2, \omega \oplus \omega \oplus \omega)$

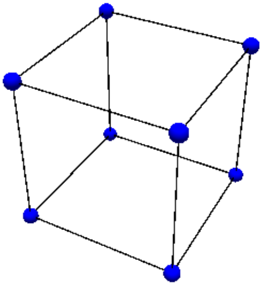
- The 1st circle rotates the 1st sphere, fixing the other two;
- The 2nd circle rotates the 2nd sphere, fixing the other two;
- The 3rd circle rotates the 3rd sphere, fixing the other two.

$$H = (h_1, h_2, h_3) : S^2 \times S^2 \times S^2 \to \mathbb{R}^3$$

8 Fixed Points.

Moment Map Image

$$H = (h_1, h_2, h_3) : S^2 \times S^2 \times S^2 \to \mathbb{R}^3$$



Hamiltonian Actions

Convexity Theorem - Atiyah, Guillemin and Sternberg '82

- (M, ω) Compact, connected, symplectic Manifold.
- $\mathbb{T}^k \curvearrowright (M, \omega)$ Hamiltonian ($\mathbb{T}^k = \overbrace{S^1 \times \cdots \times S^1}^{k}$)
- $H : M \to \mathbb{R}^k$ - Moment Map

  $H(M)$ is a convex polytope of dimension k

Toric Actions

When $\dim(\mathbb{T}^k) = k = \frac{\dim(M)}{2}$

(M, ω, H) - **Toric Manifold**

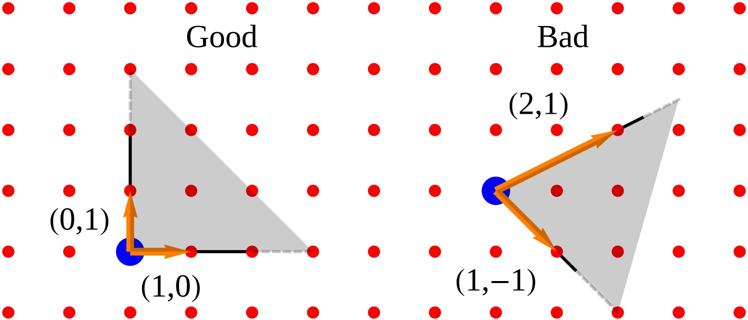
$\Delta := H(M)$ - **Delzant Polytope**

Delzant Polytopes

1. Δ is **simple**: each vertex is the intersection of exactly n edges.
2. Δ is **rational**: the edges can be generated by integer vectors.
3. Δ is **smooth**: at each vertex v we can choose vectores $u_1, \ldots, u_n$ that generate the edges through v, forming a basis of $\mathbb{Z}^n$:

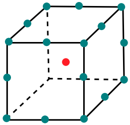
$$\det \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix} = \pm 1$$

Delzant Polytopes

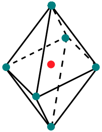


$$\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3 \neq 1$$

Delzant Polytopes



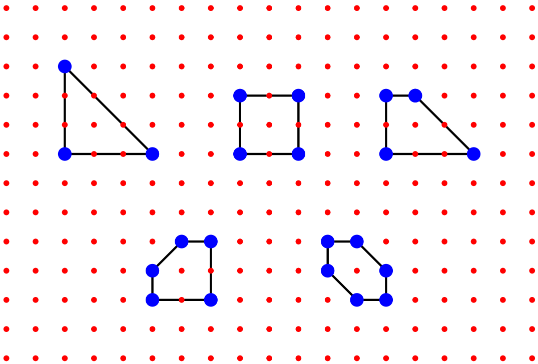
Delzant



Non-Delzant

Relations to combinatorics...

Some Special Polygons

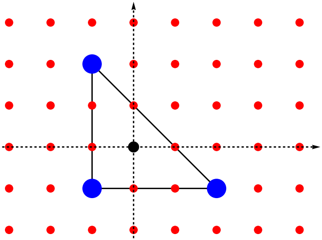


Sum of the integer length of the edges

$+$ number of vertices $=$ **12**

Some Special Polygons

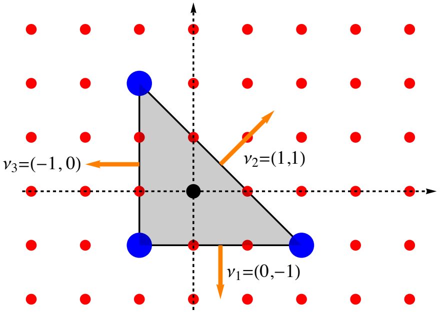
Let's see for instance the polytope



- A unique interior integer point;
- All vertices are integer points

Some Special Polygons

The shaded region



can be written as an intersection of half-spaces of the form:

$$\begin{cases} \langle (x,y), \nu_1 \rangle = \langle (x,y), (0,-1) \rangle = -y & \leq \mathbf{1} \\ \langle (x,y), \nu_2 \rangle = \langle (x,y), (1,1) \rangle = x+y & \leq \mathbf{1} \\ \langle (x,y), \nu_3 \rangle = \langle (x,y), (-1,0) \rangle = -x & \leq \mathbf{1} \end{cases}$$

Reflexive Polytopes

These Polygons are examples of **reflexive** polytopes

Reflexive Polytopes (Batyrev '94)

Δ - n-dimensional polytope. Δ is **reflexive** if

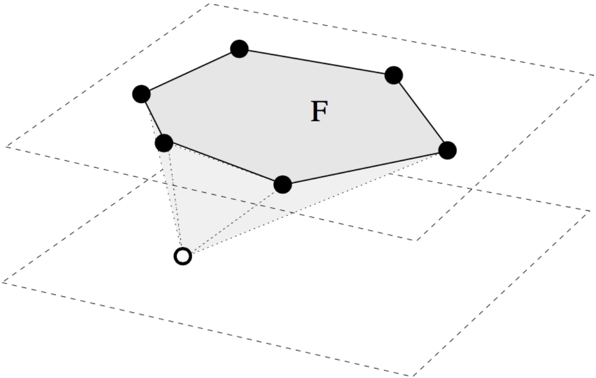
$$\Delta = \bigcap_{i=1}^{k} \{x \in \mathbb{R}^n \mid \langle x, \nu_i \rangle \leq \mathbf{1}\},$$

$\nu_i \in \mathbb{Z}^n$ - exterior (primitive) normal vector to the hyperplanes

$$H_i := \{x \in \mathbb{R}^n \mid \langle x, \nu_i \rangle = 1\}, i = 1, \ldots, k.$$

Reflexive Polytopes

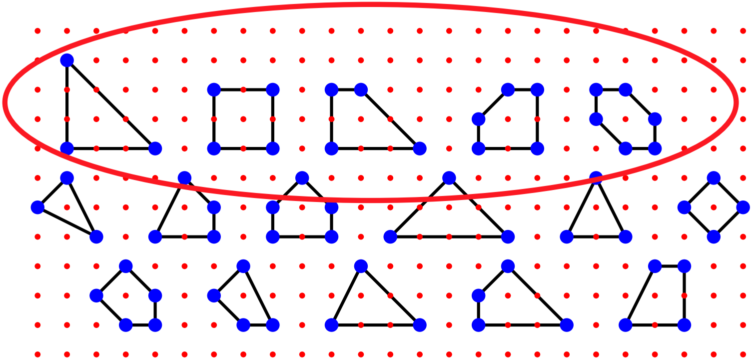
Δ is reflexive $\Leftrightarrow$ Given any facet F there are no integer points between the hyperplane generated by F and the parallel hyperplane through the origin.



Some Refelxive Polytopes are special:

Delzant Reflexive Polytopes

Among the 16 reflexive polygons there are 5 that are Delzant.



Delzant Reflexive Polytopes and the number 12

Theorem (Poonen & Rodriguez-Villegas '00)

Δ - Delzant reflexive polygon.

$$\sum_{e \in E} l(e) + |V| = \mathbf{12}.$$

where E is the edge set of Δ.

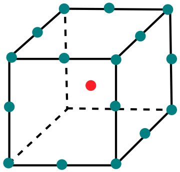
Delzant Reflexive Polytopes of dim 3 and the number 24

Theorem(Dais, Haase-Nill-Paffenholz)

Δ - Delzant Reflexive polytope of dimension 3.

$$\sum_{e \in E} l(e) = \mathbf{24}.$$

Example: The reflexive cube



Generalization

Theorem(G.-von Heymann-Sabatini, '15)

- Δ - *Delzant Reflexive Polytope* of dimension n.
- $f = (f_0, \ldots, f_n)$ - f-vector of Δ.
- E - Edge set of Δ. Then

$$\sum_{e \in E} l(e) = 12\, f_2 + (5 - 3n)\, f_1 .$$

Generalization

Theorem(G.-von Heymann-Sabatini, '15)

- Δ - *Delzant Reflexive Polytope* of dimension n.
- $f = (f_0, \ldots, f_n)$ - f-vector of Δ.
- E - Edge set of Δ. Then

$$\sum_{e \in E} l(e) = 12\, f_2 + (5 - 3n)\, f_1.$$

- If $n = 2$

$$\sum_{e \in E} l(e) = 12\, f_2 - \ f_1 = 12 - f_0$$

and then

$$\sum_{e \in E} l(e) + f_0 = 12.$$

Theorem(G.-von Heymann-Sabatini, '15)

- Δ - *Delzant Reflexive Polytope* of dimension n.
- $f = (f_0, \dots, f_n)$ - f-vector of Δ.
- E - Edge set of Δ.

$$\sum_{e \in E} l(e) = 12\, f_2 + (5 - 3n)\, f_1.$$

- If $n = 3$

$$\sum_{e \in E} l(e) = 12\, f_2 - 4\, f_1$$

Euler Relation $f_0 - f_1 + f_2 = 2$,

$$\sum_{e \in E} l(e) = 12\, f_2 - 4\, f_1 = 24 - 12 f_0 + 8 f_1.$$

Generalization

- If $n = 3$

$$\sum_{e \in E} l(e) = 12\, f_2 - 4\, f_1 = 24 - 12 f_0 + 8 f_1$$

For simple polytopes, we have $3\, f_0 = 2\, f_1$ and then

$$\sum_{e \in E} l(e) = 12\, f_2 - 4\, f_1 = 24 - 12 f_0 + 8 f_1 = 24$$

The End!

Thanks!