Mathematics for Blood Flow Simulations

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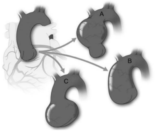
Motivation - cardiovascular diseases

Atherosclerosis

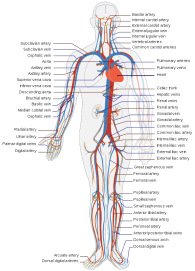
Aneurysms

Bicuspid Aortic Valves

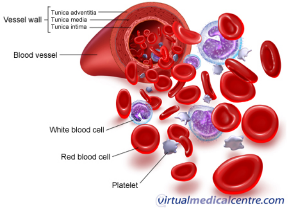


Aortic dilation

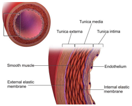
Modeling



(a) *medicalnoises.com*



(b) *myvmc.com*



(c) The blood vessel structure. Font: from `https://en.wikipedia.org/wiki/Tunica_intima`

Blood is a suspension of particles in the plasma (92% of water)

- Red Blood Cells (RBC) - $6 - 8\ \mu m$/ $4 - 6 \times 10^6$ per mm^3
- White Blood Cells - $8 - 18\ \mu m$ / $4 - 10 \times 10^3$ per mm^3
- RBC - 45% of blood volume.

Continuum mechanics - flow balance equations at constant temperature and density

- Balance of linear momentum

$$m\vec{a} = \vec{F}$$

- $\rho \frac{Du}{\partial t} \quad = \quad F_{\text{contact and surface}} + F_{\text{volume}}$
- $\rho(\frac{\partial u}{\partial t} + u \cdot \nabla u) = F_{\text{contact and surface}} + F_{\text{volume}}$
- $\rho(\frac{\partial u}{\partial t} + u \cdot \nabla u) = \nabla \cdot T + F_g$

what is T?

$$T = 2\mu(\dot{\gamma})D(\mathrm{u}) - pI$$

$p \rightarrow$ pressure (scalar function) with $\dot{\gamma} = \sqrt{D(\mathrm{u}) : D(\mathrm{u})}$

Non-Newtonian rheology

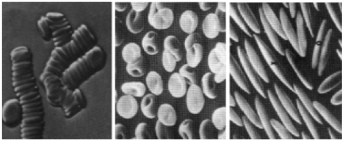
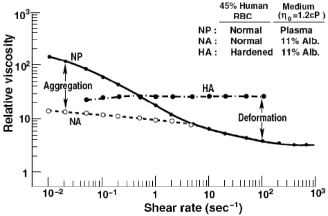


Figure: RBC form three-dimensional structures at low shear rates mainly under pathological conditions.

- $\mu(\dot{\gamma}) = \mu_\infty + \frac{\mu_0 - \mu_\infty}{(1+(\lambda\dot{\gamma})^b)^{\frac{1-n}{b}}}$ $\rightarrow$ Carreau-Yasuda
  - $\mu_0 = 6.57 \cdot 10^{-2}\ Pa.s$,
  - $\mu_\infty = 4.47 \cdot 10^{-3}\ Pa.s$
  - $n = 0.34$, $b = 1.76$, $\lambda = 10.4\ s$
  - $U_0 = 0.0662\ m/s$, $\rho = 1059\ Kg/m^3$.
- Power Law, Carreau, Cross, etc,...



- *Robertson, Sequeira, Kameneva, Hem. Fl. Mod. An. Sim., 2008.*
- *Fahraeus* - 1929.
- *Chien* - 1970.

Viscoelastic effects may also appear [Robertson, Sequeira, Kameneva, Hem. Fl. Mod. An. Sim., 2008]

Continuum mechanics - flow balance equations at constant temperature and density

- Balance of mass (mass conservation)

$$\frac{d\,m}{d\,t} \quad = \quad 0$$

- $\frac{\partial \rho}{\partial t} + div(\rho u) = 0$

The resulting coupled system: (generalized) Navier-Stokes equations.

Configurations with meaningful boundary conditions (for constant viscosity) were studied in [Heywood, Rannacher, Turek, 1996].

Continuum mechanics - flow balance equations at constant temperature and density

- Balance of mass (mass conservation)

$$\frac{d\,m}{d\,t} = 0$$

- $\frac{\partial \rho}{\partial t} + div(\rho u) = 0$
- $div(u) = 0$, (density ρ is constant)

The resulting coupled system: (generalized) Navier-Stokes equations.

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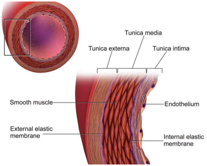
Wall modelling

- Balance of linear momentum

$$m\vec{a} = \vec{F}$$



- $\rho_w \frac{\partial^2 \eta}{\partial t^2} \quad = \quad div(P)$

$$P = (I + \nabla\eta)S(\eta)$$

with

$$S = \lambda tr(E)I + 2\mu_s E$$

and

$$E = [\nabla\eta + (\nabla\eta)^T + (\nabla(\eta))^T\nabla(\eta)]$$

(Saint-Venant Kirchhoff material)

For more realistic models see [Holzapfel, Ogden, 2010]

Coupled Model: Fluid + Wall Structure (ALE)

Intro in [Fernandez, Formaggia, Gerbeau, Quarteroni, chap 3, Card. Math., FQV eds, 2009]

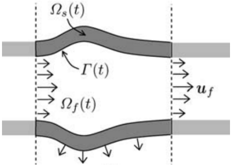
$$
\begin{cases}
\rho(\frac{\partial u}{\partial t} + (u - w)\cdot\nabla u) = \nabla\cdot T + F_g & \text{in } \Omega_f \\
div\ u = 0 & \text{in } \Omega_f, \\
\rho_w \frac{\partial^2 \eta}{\partial t^2} = div(P) & \text{in } \Omega_s, \\
u = \frac{\partial \eta}{\partial t} & \text{on } \Gamma \\
P\cdot n = T\cdot n & \text{on } \Gamma \\
u = g & \text{on } \Gamma_{in} \\
T\cdot n = h & \text{on } \Gamma_{out} \\
\text{b.c.s on } \eta & \text{on } \partial\Omega_s \setminus \Gamma.
\end{cases}
$$

$T = \mu(\dot\gamma)(\nabla \mathrm{u} + (\nabla \mathrm{u})^T) - p\, I_d = 2\mu(\dot\gamma)D(\mathrm{u}) - p\, I_d$

$p \rightarrow$ pressure (scalar function)

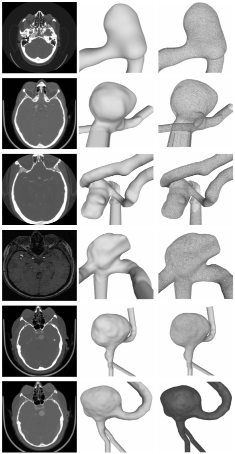
w is the time derivative of the displacement of points in Ω_f computed as an extension of η.

$P = (I + \nabla\eta)(\lambda tr(E)I + 2\mu_s E)$

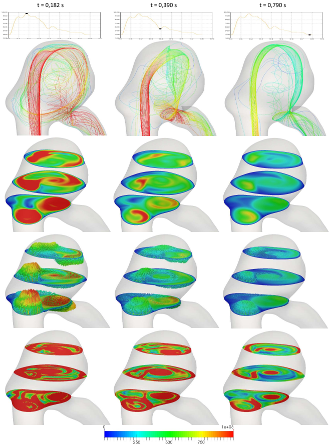


- Numerical methods [Fernandez, Gerbeau,...]
- Mathematical Analysis [Grandmont, Beirão da Veiga, Maday,..]

Example: Flow structures in brain aneurysms



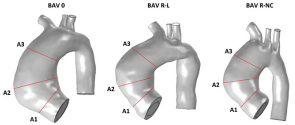
(a) Computational domains



(b) row 1: Velocity streamlines; row 2: velocity magnitude; row 3: velocity vectors; row 4: vorticity magnitude. Incremental Pressure Correct. Scheme with FEnics. See evolutive solution: 

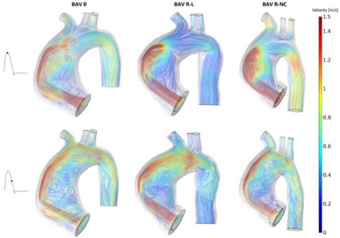
with Iolanda Velho, Ricardo Pereira, Adélia Sequeira, in I. Velho, **Study of Cerebral Aneurysms Through Experimental Biomarkers and Computational Hemodynamics**, PhD thesis, IST, 2019

Example: Fluid Structure Interaction in Dilated Aortas

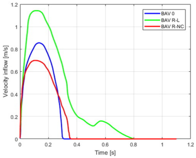


(a) Computational domains

(b) Bicuspid aortic valves



(c) Velocity streamlines. WSS here



(d) Patient specific inflow (EchoDoppler

- Absorbing outflow boundary conditions: $P^{n+1} = RQ^n$ [Janela, Moura, Sequeira, 2010]
- Arbitrarian Lagrangian Eulerian approach [Quarteroni, Formaggia, 2009]

with Diana Oliveira, Sílvia Aguiar Rosa, Rui Cruz Ferreira, Ana Figueiredo Agapito, Adélia Sequeira, **Bicuspid aortic valve aortopathies: An hemodynamics characterization in dilated aortas**, Computer Methods in Biomechanics and Biomedical Engineering, 22(8), 815 - 826 (2019).

What to use as boundary conditions in the artificial boundaries?

Using surrogate models → Geometric multiscale:

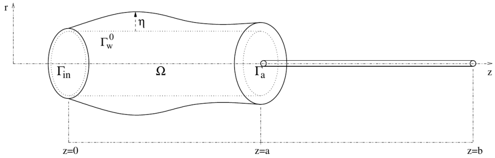


Figure: 3D−1D coupling [Quarteroni, Formaggia, 2009]

- 1D model solves for the the flow rate Q and area A on the cross section located at z.
- The pressure $\bar{p}$ at z is a function of A.
- Continuity of the flux $\int_{\Gamma_a} \mathbf{u} \cdot n\, ds = Q(a)$
- Continuity of normal stresses $\frac{1}{\Gamma_a} \int_{\Gamma_a} (p - 2\mu D(\mathbf{u})) \cdot n\, ds = \bar{p}(a) + \frac{\rho}{2} \bar{\mathbf{u}}(a)^2$

Energy estimates (total pressure) in [Formaggia, Moura, Nobile, 2007]

Theoretical Frame: Control Problem - Existence Result

Minimize

$$J(u, g) = w_1 \int_{\Omega_{part}} |u - u_d|^2 dx + w_2 \int_{\Gamma_{in}} |g|^2 ds + w_3 \int_{\Gamma_{in}} |\nabla_s g|^2 ds$$

[$\Omega_{part} = \cup_{i=1}^{m} s_i$ where S_i are cross sections of the "vessel" domain]

subject to

$$\begin{cases} -\nu \triangle u + u \cdot \nabla u + \nabla p = f & \text{in } \Omega, \\ div\, u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_{wall} \\ u = g & \text{on } \Gamma_{in} \\ \mu \frac{du}{dn} - np = 0 & \text{on } \Gamma_{out}. \end{cases}$$

$$g \in \mathcal{U}, \quad u \in V_{\Gamma_{wall}} = \left\{ v \in H^1(\Omega) : \gamma_{\Gamma_{wall}} v = 0,\ div\, v = 0 \right\}$$

$$\mathcal{U} = \left\{ g \in H_0^1(\Gamma_{in}) \mid \|g\|_{H_0^1(\Gamma)} \leq \rho \right\} \subset \mathcal{U}_0$$

$$\mathcal{U}_0 = \left\{ g \in H_0^1(\Gamma_{in}) : \text{s.t. NSEq have a unique weak solution} \right\}.$$

Research Topics

- Control Approach for boundary estimation in blood flow model.
- Parameter estimation in Fluid-Structure Interaction Models.