Equilibria Computation in Algorithmic Game Theory

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14 March 2024

Algorithmic game theory

- many problems in computer science involve **interactions** between multiple self-interested **agents**;
  - traffic and vehicle routing;
  - resource allocation;
  - online advertising;
- economics and game theory: useful models, **analytical** study;
- algorithmic game theory: brings a **computational** perspective.

Algorithmic game theory

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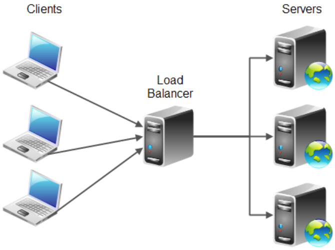
Contributions from Computer Science:

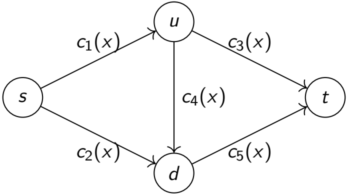
- worst-case guarantees;
- approximation bounds;
- computational complexity.

- a set of **players** N: each $i \in N$ has a **strategy set** S_i;
- **costs** $C_i(s_i, \mathbf{s}_{-i})$ depending on the joint strategy of all players;
- **Pure Nash equilibrium** (PNE): a strategy profile where no player gains by deviating.

$$C_i(s_i, \mathbf{s}_{-i}) \leq C_i(s_i', \mathbf{s}_{-i}) \quad \text{for all} \quad i \in N, s_i' \in S_i.$$

Example: Congestion games





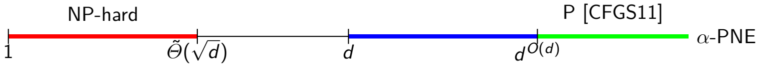
Results for congestion games

- in the **unweighted** case, PNE always exist and can be found by **best-response sequences** [Ros73].
- when players have **weights**, PNE need not exist [LO01, GMV05, FKS05].
- a more general notion: α-**approximate** equilibrium (α-PNE), for $\alpha \geq 1$;

$$C_i(s_i, \mathbf{s}_{-i}) \leq \alpha \cdot C_i(s_i', \mathbf{s}_{-i}) \quad \text{for all} \quad i \in N, s_i' \in S_i.$$

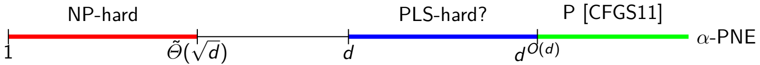
State of the art

- **polynomial** congestion games: costs are polynomials of degree d.
- **nonexistence** of exact or α-equilibria for low α. [FKS05, HKS14, CGG⁺23]
- **NP-hardness** of the exact or α-approximate decision problem for low α. [CGG⁺23]
- **existence** of α-equilibria for large α. [CF19]
- **efficient computation** of α-equilibria for even larger α. [CFGS11]

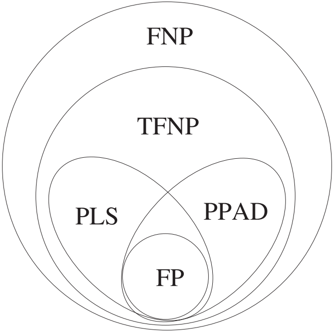


Future directions

- can we **narrow** the existence gap between $\sqrt{d}$ and d?
- what about the complexity of finding approximate equilibria when we **know** they exist?
- PLS = Polynomial Local Search problems.



Future directions [Rou16]



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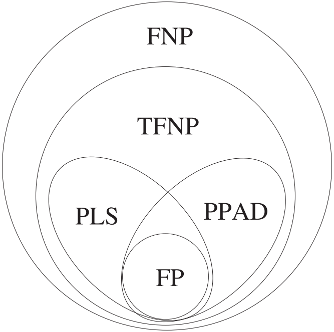

Considering random perturbations on the costs:

- What is the **average-case** complexity of best-responses?
- What is the **smoothed** complexity of best-responses?

Conclusion

- Congestion games: a natural framework to study competition.
- Game theory: Nash equilibria as a predictive concept.
- When does a game have Nash equilibria?
- How to compute Nash equilibria?

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Thank you for your attention!

`https://diogopocas1991.gitlab.io`

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