The Sphere Packing Problem

Diogo Oliveira e Silva

MMAC Day 2024

π-day 2024

What fraction of Euclidean space $\mathbb{R}^d$ can be covered by congruent balls with disjoint interiors?

Precise formulation

A **sphere packing** is a union of the form

$$\mathcal{P} = \bigcup_{x \in X} B_1^d(x)$$

The **upper density** of $\mathcal{P}$ is

$$\Delta_{\mathcal{P}} := \limsup_{R \to \infty} \frac{\mathrm{Vol}(\mathcal{P} \cap B_R^d)}{\mathrm{Vol}(B_R^d)}$$

The number we are looking for is

$$\Delta_d := \sup_{\substack{\mathcal{P} \subset \mathbb{R}^d \\ \text{sphere packing}}} \Delta_{\mathcal{P}}$$

Two special cases: **lattice** packing, **periodic** packing.

What is known?

- $d = 1$: $\Delta_1 = 1$, trivial
- $d = 2$: Thue* (1892), Fejes Tóth (1942)

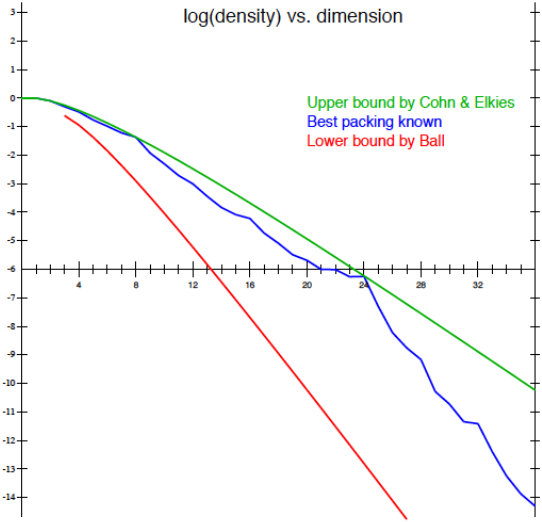
$$\Delta_2 = \frac{\pi}{\sqrt{12}} \approx 0.90690...$$

- $d = 3$: Hales (1998)

$$\Delta_3 = \frac{\pi}{\sqrt{18}} \approx 0.74048...$$

- $d = 8$: **Viazovska** (π-day 2016, Fields Medal 2022)
- $d = 24$: Cohn–Kumar–Miller–Radchenko–Viazovska (2016)
- Upper bounds
  - Kabatyanskii–Levenshtein (1978): $\Delta_d \leq 2^{(-0.599...+o(1))d}$
- Lower bounds after Minkowski (1905): $\Delta_d \geq 2 \cdot 2^{-d}$
  - Ball (1992): $\Delta_d \geq 2d \cdot 2^{-d}$
  - Venkatesh (2013): $\Delta_d \gtrsim d \log\log d \cdot 2^{-d}$
  - Campos–Jenssen–Michelin–Sahasrabudhe (Dec 2023): $\Delta_d \gtrsim d \log d \cdot 2^{-d}$

Low dimensions ($1 \leq d \leq 36$)



Upper bounds on Sphere Packings

Theorem (Cohn–Elkies, 2003)

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a Schwartz function, and $r > 0$ be such that:

- $f(0) = \widehat{f}(0) > 0$;
- $\widehat{f}(\xi) \geq 0$ *for every* $\xi \in \mathbb{R}^d$;
- $f(x) \leq 0$ *for* $|x| \geq r$.

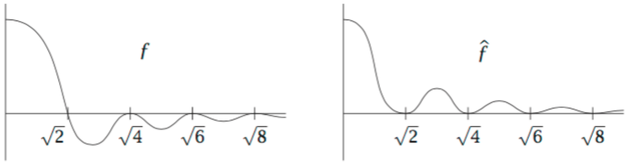
Then the following upper bound for the sphere packing constant holds:

$$\Delta_d \leq \mathrm{Vol}(B^d_{r/2})$$

Proof. (For lattice packings only) Density equals $\frac{\mathrm{Vol}(B^d_{r/2})}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)}$, so enough to show denominator is at least 1. This follows from Poisson:

$$f(0) \geq \sum_{x\in\Lambda} f(x) = \frac{1}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)} \sum_{\xi\in\Lambda^*} \widehat{f}(\xi) \geq \frac{\widehat{f}(0)}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)}$$

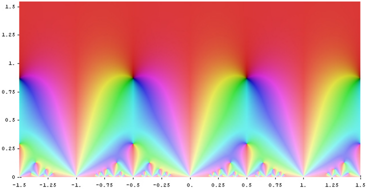
How does the magic function look like in $\mathbb{R}^8$?

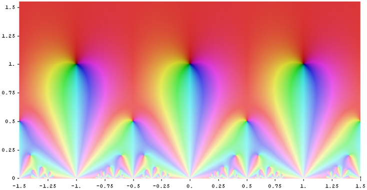


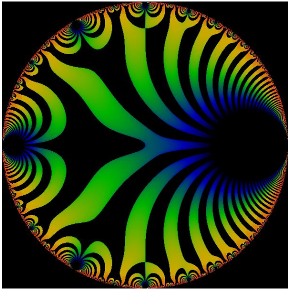
- $f : \mathbb{R}^8 \to \mathbb{R}$ radial, Schwartz
- $f(0) = \widehat{f}(0) = 1$
- $\widehat{f}(\xi) \geq 0$ for all $\xi \in \mathbb{R}^8$
- $f(x) \leq 0$ for $|x| \geq \sqrt{2}$
- The values $f(r)$ and $\widehat{f}(r)$ vanish whenever $r^2 \in 2\mathbb{Z}_{>0}$

The Cohn–Elkies optimisation problem is easy to solve if $d = 1$, but the "luck" seemed to stop there...

Modular forms (E_4, E_6, $\operatorname{Re}\Delta$)







The magic function in $\mathbb{R}^8$

$$\boxed{f(x) = \frac{\pi i}{8640} a(x) + \frac{i}{240\pi} b(x)}$$

$+1$ eigenfunction is

$$a(r) = -4\sin^2(\tfrac{\pi r^2}{2}) \int_0^{i\infty} \phi_0(\tfrac{-1}{z}) z^2 e^{\pi i r^2 z}\,\mathrm{d}z,$$

$$\phi_0 = \frac{(E_2 E_4 - E_6)^2}{\Delta}$$

-1 eigenfunction is

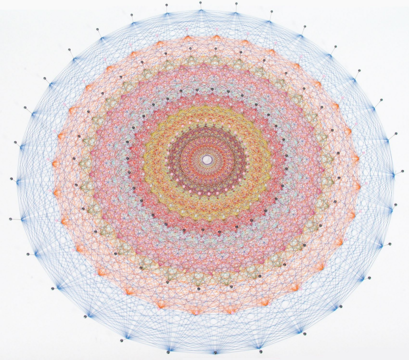
$$b(r) = -4\sin^2(\tfrac{\pi r^2}{2}) \int_0^{i\infty} \psi_I(z) e^{\pi i r^2 z}\,\mathrm{d}z,$$

$$\psi_I = 128\frac{\theta_{00}^4 + \theta_{01}^4}{\theta_{10}^8} + 128\frac{\theta_{01}^4 - \theta_{10}^4}{\theta_{00}^8}$$

The Sphere Packing Problem in dimension 8

Theorem (Viazovska, 2016)

No packing of unit balls in Euclidean space $\mathbb{R}^8$ has density greater than that of the Λ_8-lattice packing.



$$\Lambda_8 = \left\{ (x_i) \in \mathbb{Z}^8 \cup \left(\mathbb{Z} + \tfrac{1}{2}\right)^8 : \sum_{j=1}^{8} x_j \equiv 0 \pmod 2 \right\}$$