Optimizing shapes and partitions

MMAC day, 14 March 2024

Hugo Tavares
CAMGSD & Departamento de Matemática do Técnico

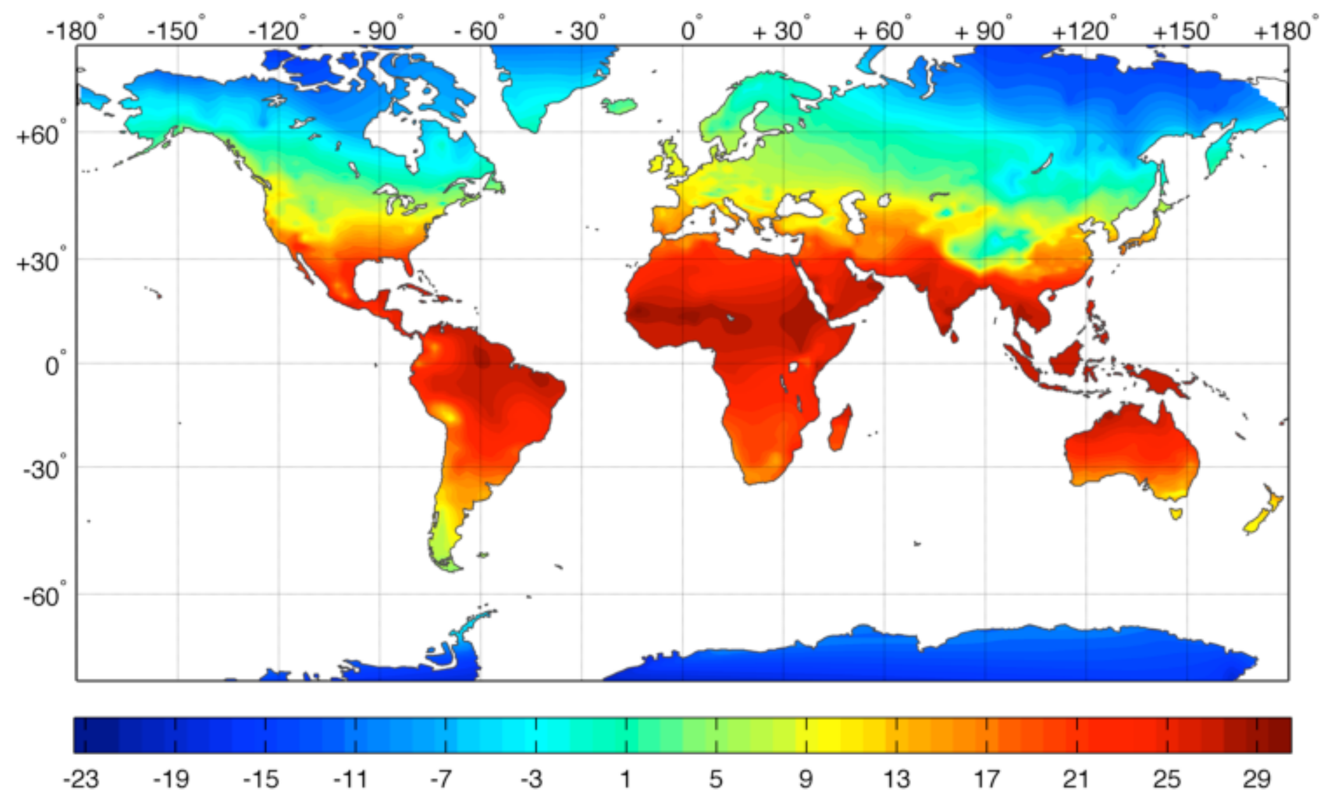
CAMGSD

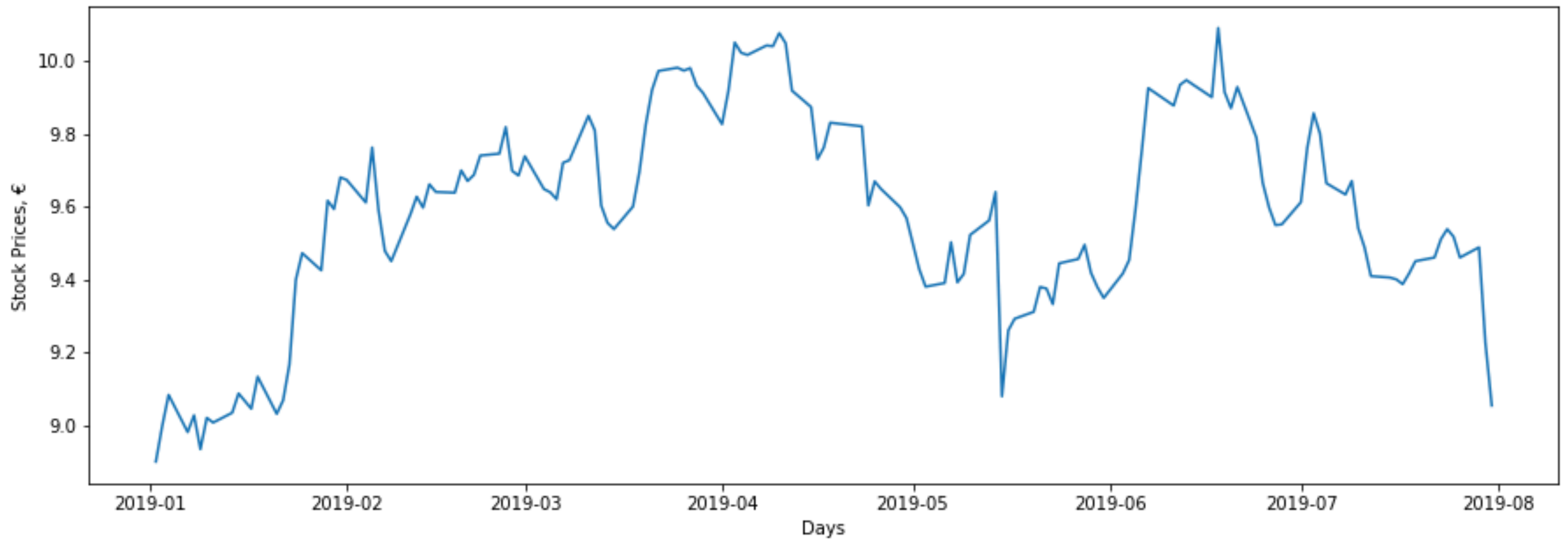
DM
DEPARTAMENTO
DE MATEMÁTICA
TÉCNICO LISBOA

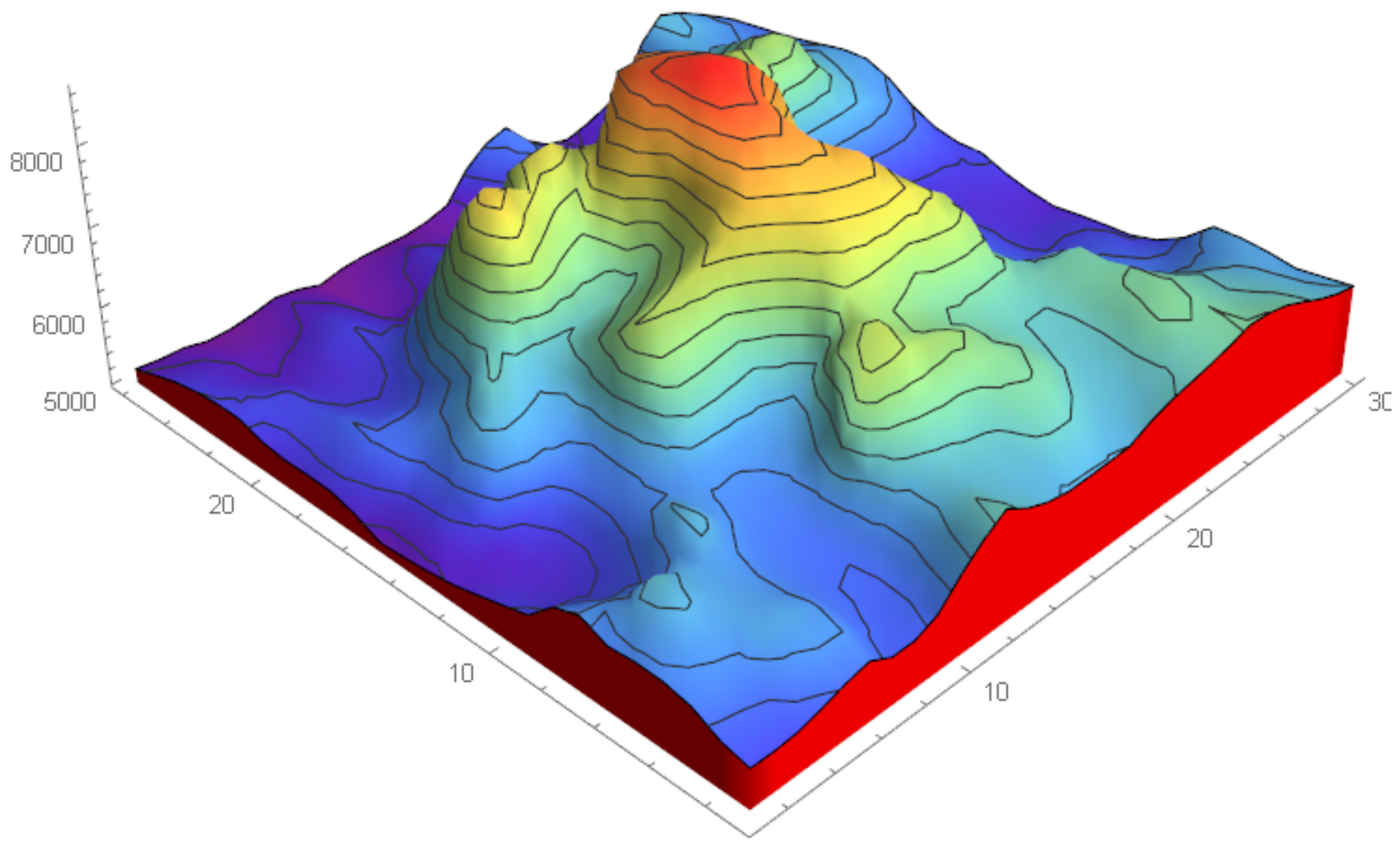
We all understand the importance of minimizing/maximizing

functions of real variables

$$f : D \subseteq \mathbb{R}^n \to \mathbb{R}$$







But not every problem fits into these formalisms. A famous example:

From all sets with a given fixed perimeter, which one has the largest area?

$$\max\{\text{area}(A) : \ A \subset \mathbb{R}^2, \text{per}(A) = a\}\,.$$

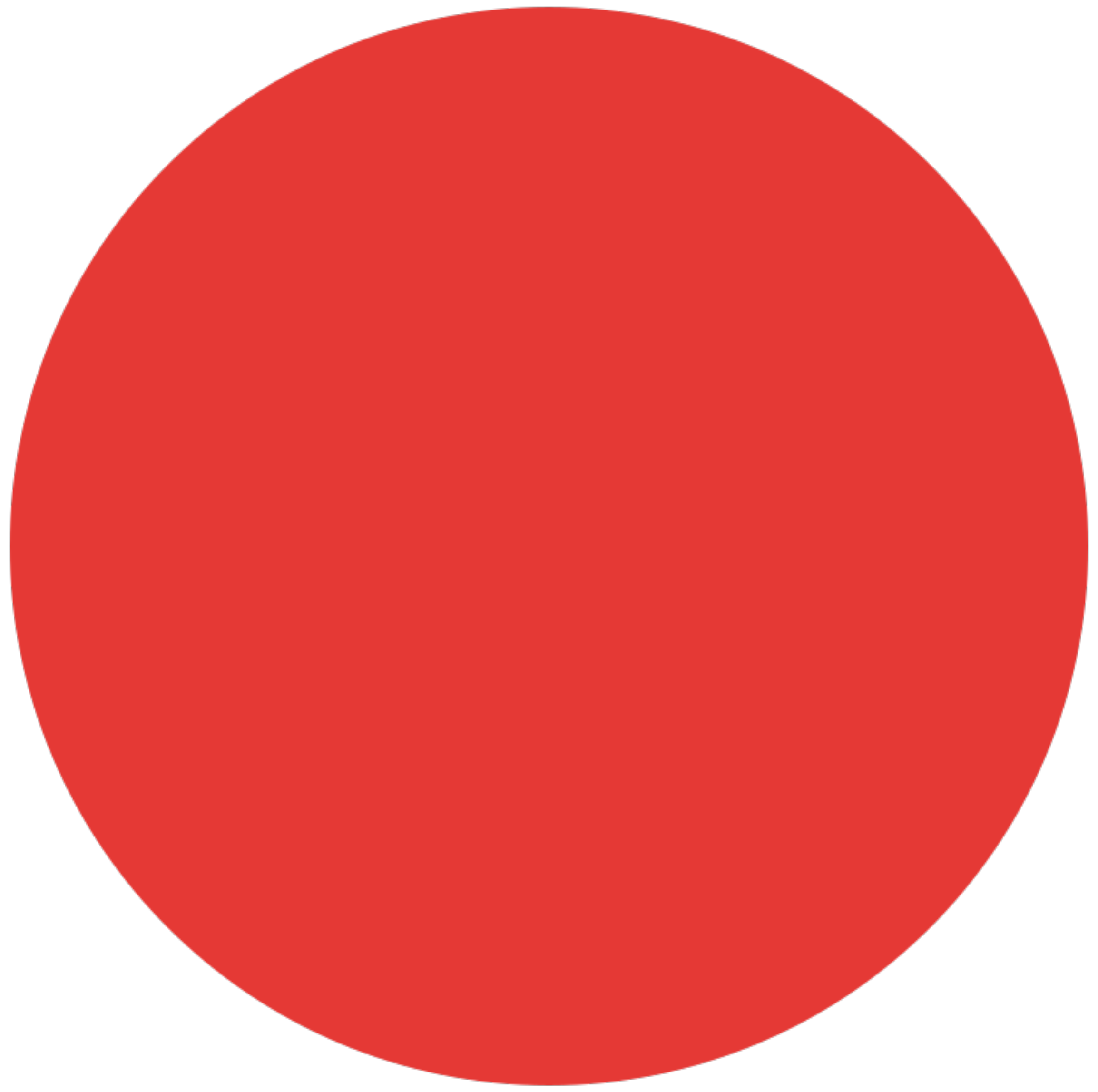
Dido's problem (IX b.C.) Dido is a legendary figure associated with Ancient Carthage (Tunisia). According to the legend, she was a refugee from a power struggle with her brother in Lebanon. She arrived with her entourage and asked a piece of land. The locals offered her as much land as could be enclosed by the hide of a bull.

But not every problem fits into these formalisms. A famous example:

From all sets with a given fixed perimeter, which one has the largest area?

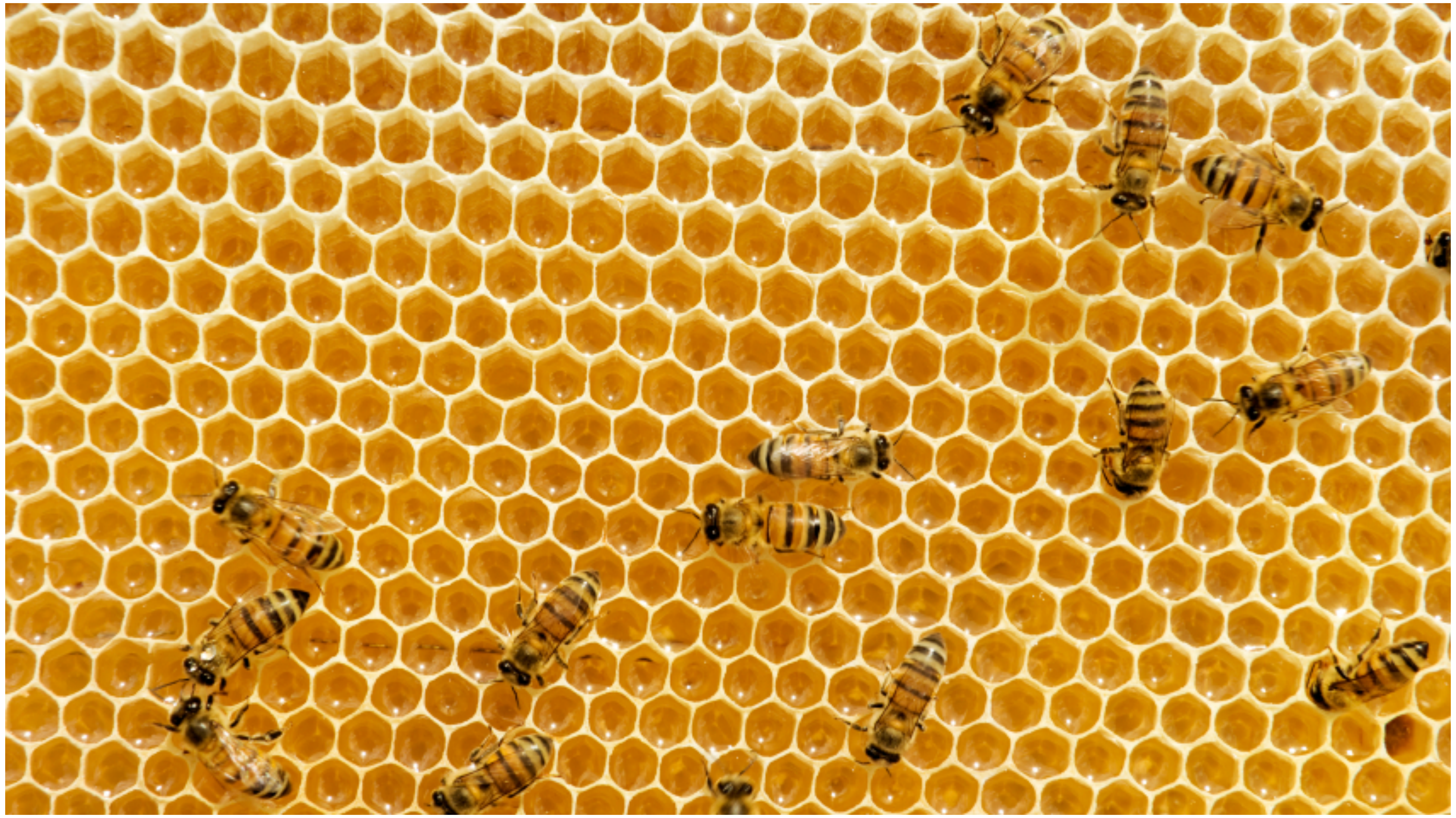
$$\max\{\text{area}(A) : \ A \subset \mathbb{R}^2, \text{per}(A) = a\} .$$

R:



Another example:

The honeycomb conjecture: among all subdivisions of the plane into regions of equal area, the regular hexagon grid is the one with least total perimeter



This conjecture comes from Ancien Rome (I-II b.C.). Proved in 1999 (!) by the mathematician Thomas Hales

Another problem: fundamental frequencies of vibration

Of all the drums with a certain area, which one has the smallest fundamental frequency?

Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Fourier series:

$$u(x,y) = \sum_{n\in\mathbb{N}} \left(a_n \cos(\sqrt{\lambda_n}ct) + b_n \sin(\sqrt{\lambda_n}ct)\right) \varphi_n(x)$$

https://en.wikipedia.org/wiki/Vibrations_of_a_circular_membrane

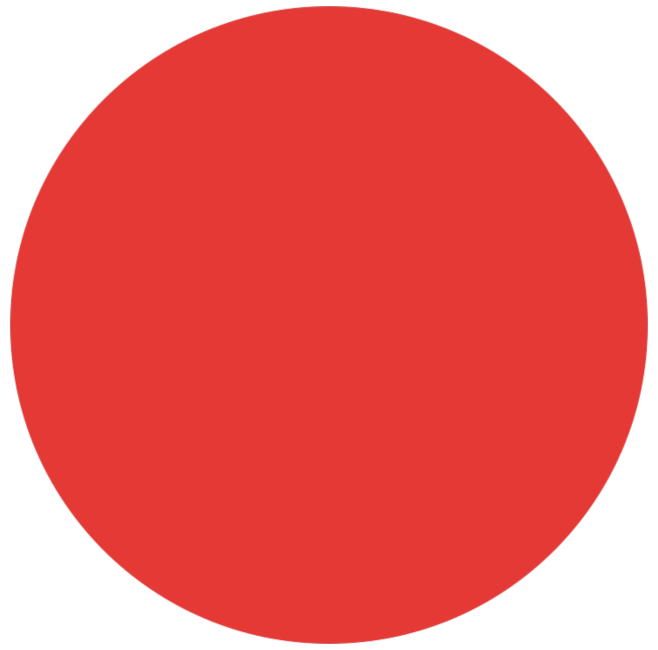
Another problem: fundamental frequencies of vibration

Of all the drums with a certain area, which one has the smallest fundamental frequency?

Given a region $\omega \subset \mathbb{R}^2$ (drum's surface), is frequencies correspond to the real numbers λ for which the following problem has a nontrivial solution

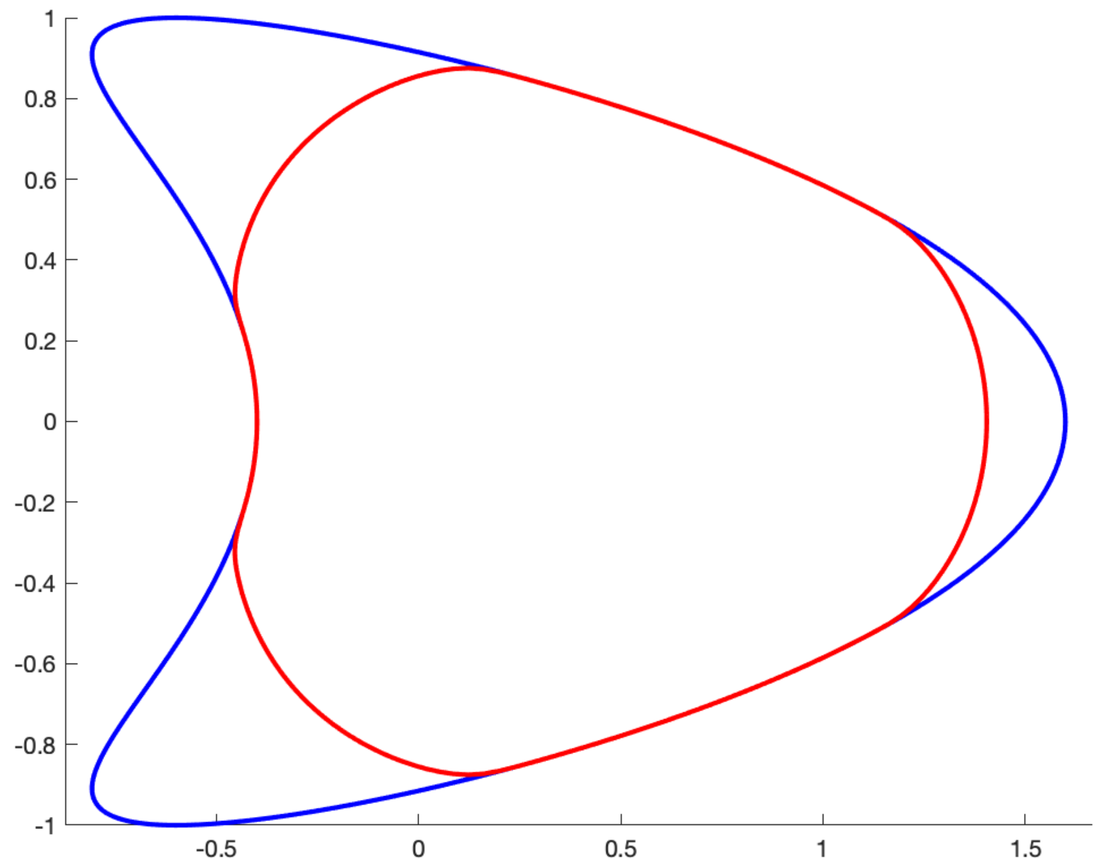
$$\begin{cases} -\Delta u(x,y) = \lambda u(x,y), & (x,y) \in \omega, \\ u(x,y) = 0, & (x,y) \in \partial\omega \end{cases} \longrightarrow \lambda_1(\omega) \leq \lambda_2(\omega) \leq , \ldots \to \infty$$

$$\min\{\lambda_1(\omega) : \ \omega \subset \mathbb{R}^2, \text{area}(\omega) = a\} .$$



Now let us add a box D, and force the set to be inside D

$$\min\{\lambda_1(\omega) : \ \omega \subset D, \text{ area}(\omega) = a\}$$

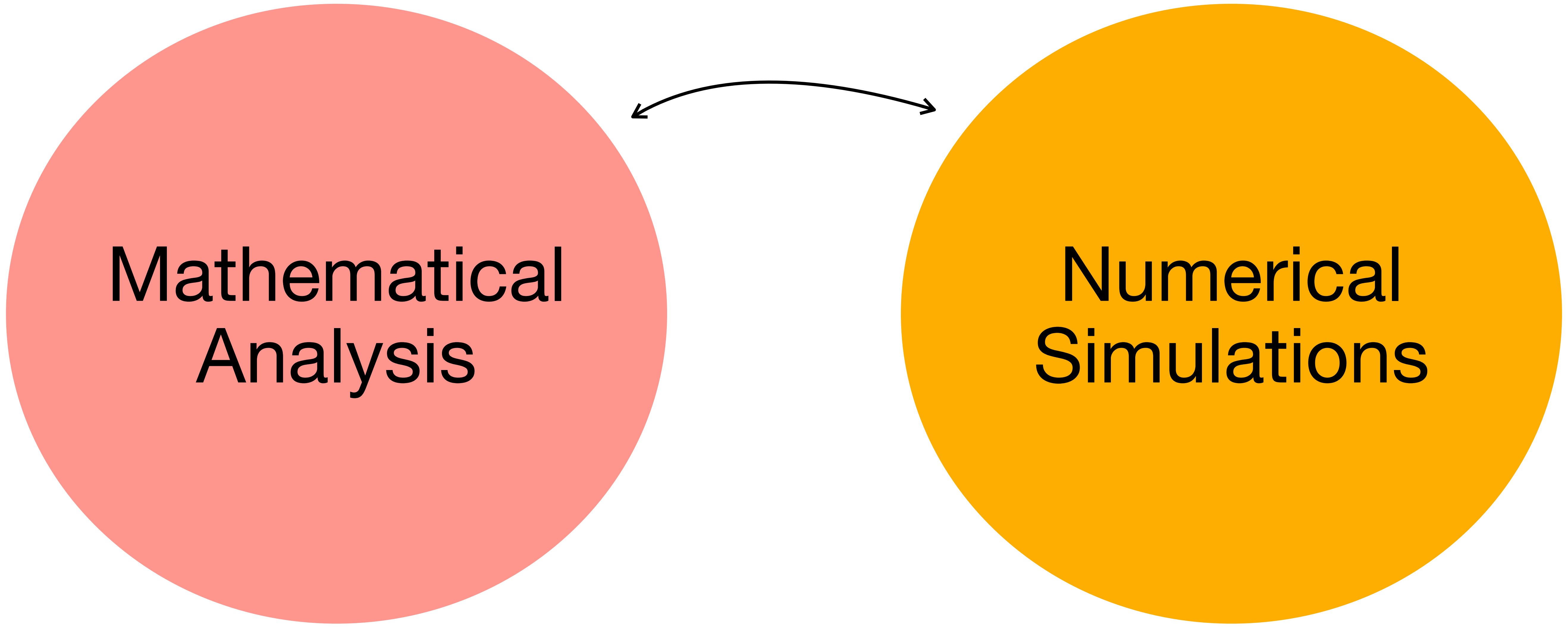


Natural questions:

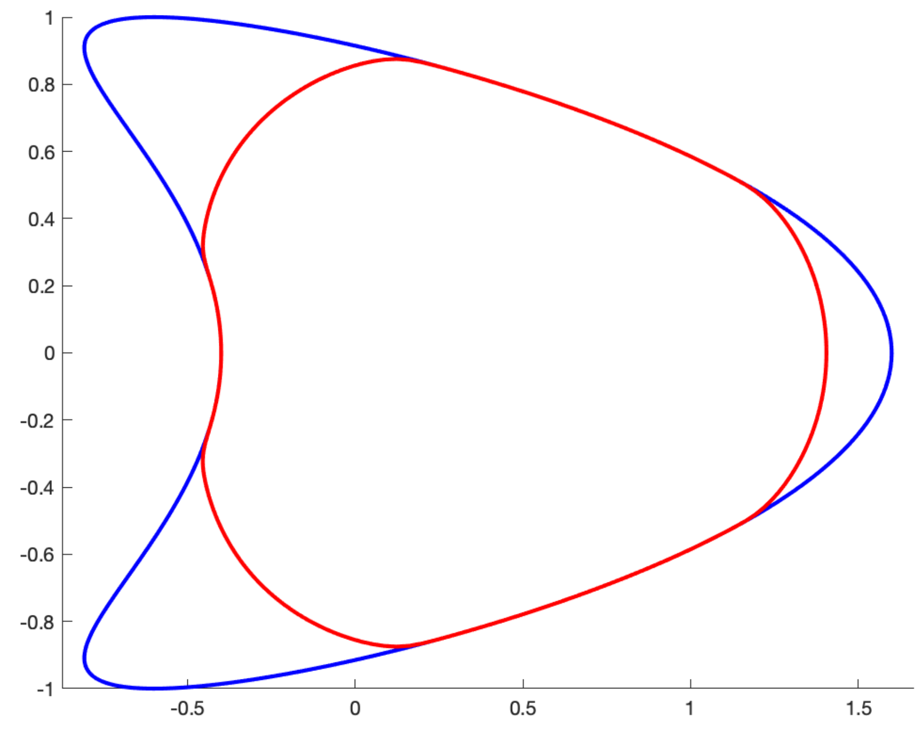
- Is there a solution? In which sense it depends on the box?
- Are the solutions explicit? Is there any pattern emerging?
- How regular are the solutions?
- In which sense are the results depending on the fact that we are minimising λ_1?

$$\min\{\lambda_1(\omega) : \ \omega \subset D, \ \mathrm{vol}(\omega) = a\}$$

Simulations by Pedro Antunes (DMIST)



Recent theoretical results (2005-…): all solutions are smooth, they intersect the boundary of the box, there are no arcs of circunference (!), …

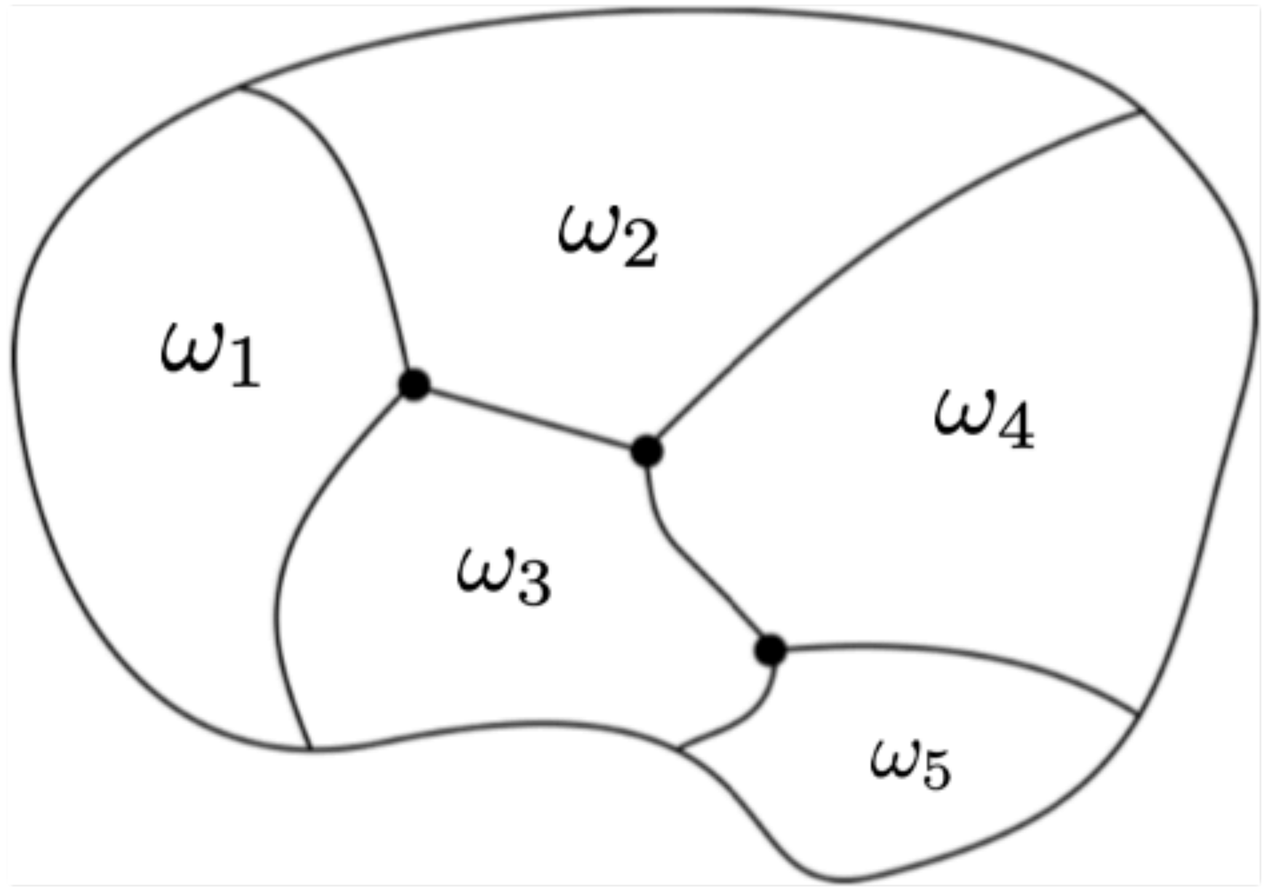


Up to this point: problems with the structure:

$$\min\left\{\Phi(\omega):\ \omega \subseteq D\ +\ \text{constraints}\right\}$$

Harder: optimal partition problems

$$\min\left\{\Phi(\omega_1,\ldots,\omega_\ell):\ \omega_i \subseteq D,\ \omega_i \cap \omega_j = \varnothing\ \forall i \neq j\right\}$$

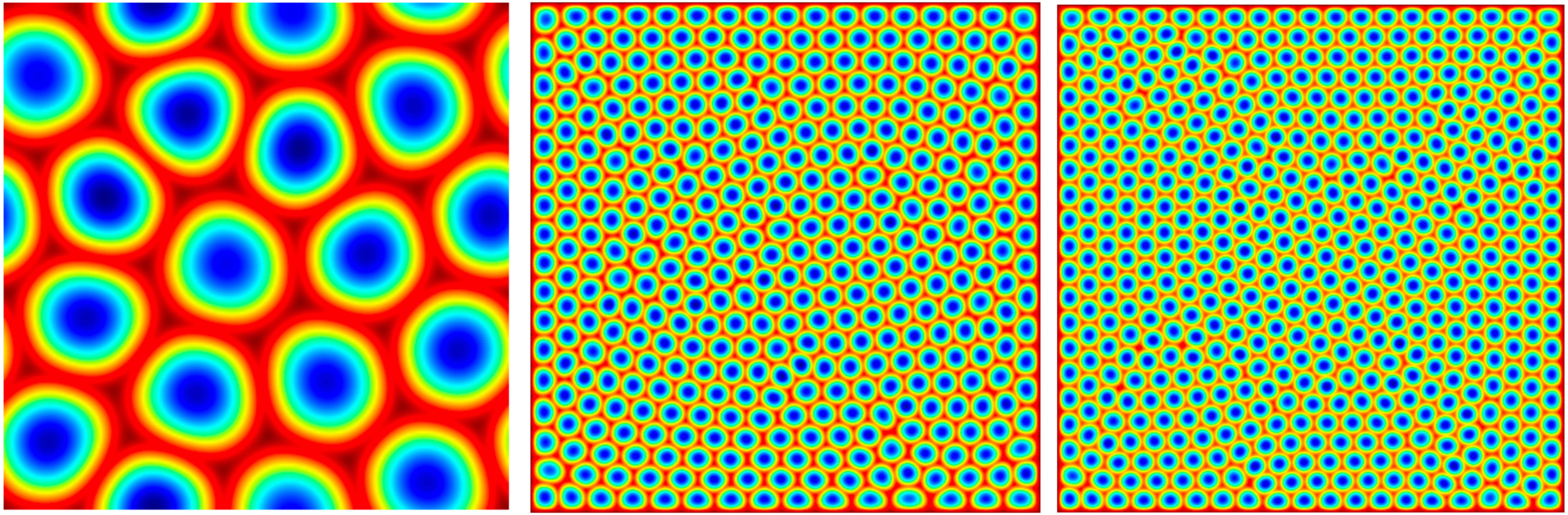


Example:

$$\Phi(\omega_1,\ldots,\omega_\ell) = \sum_{i=1}^{\ell} \lambda_1(\omega_i)$$

(Ramos-Tavares-Terracini 2016)

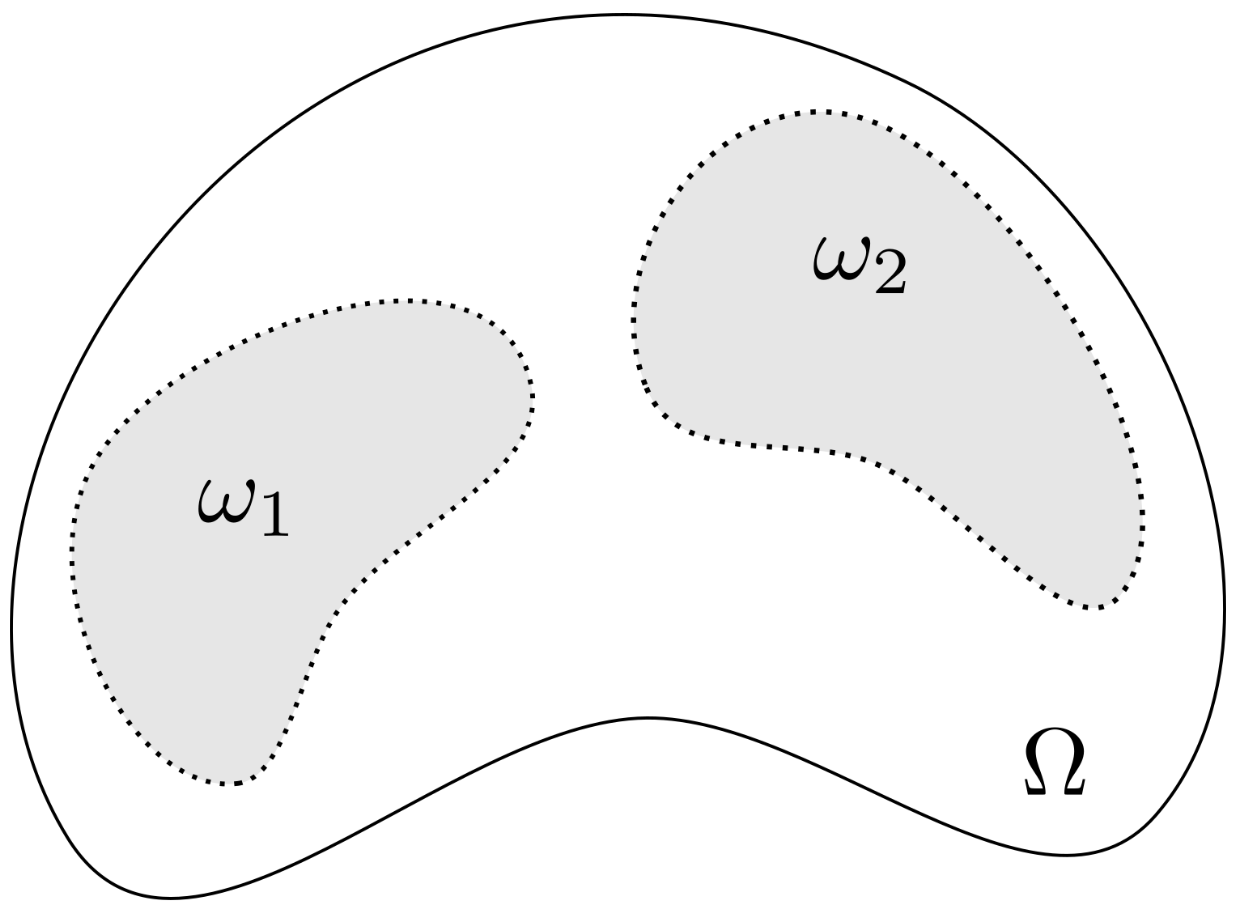
Hexagons again as $\ell \to \infty$? Open problem!



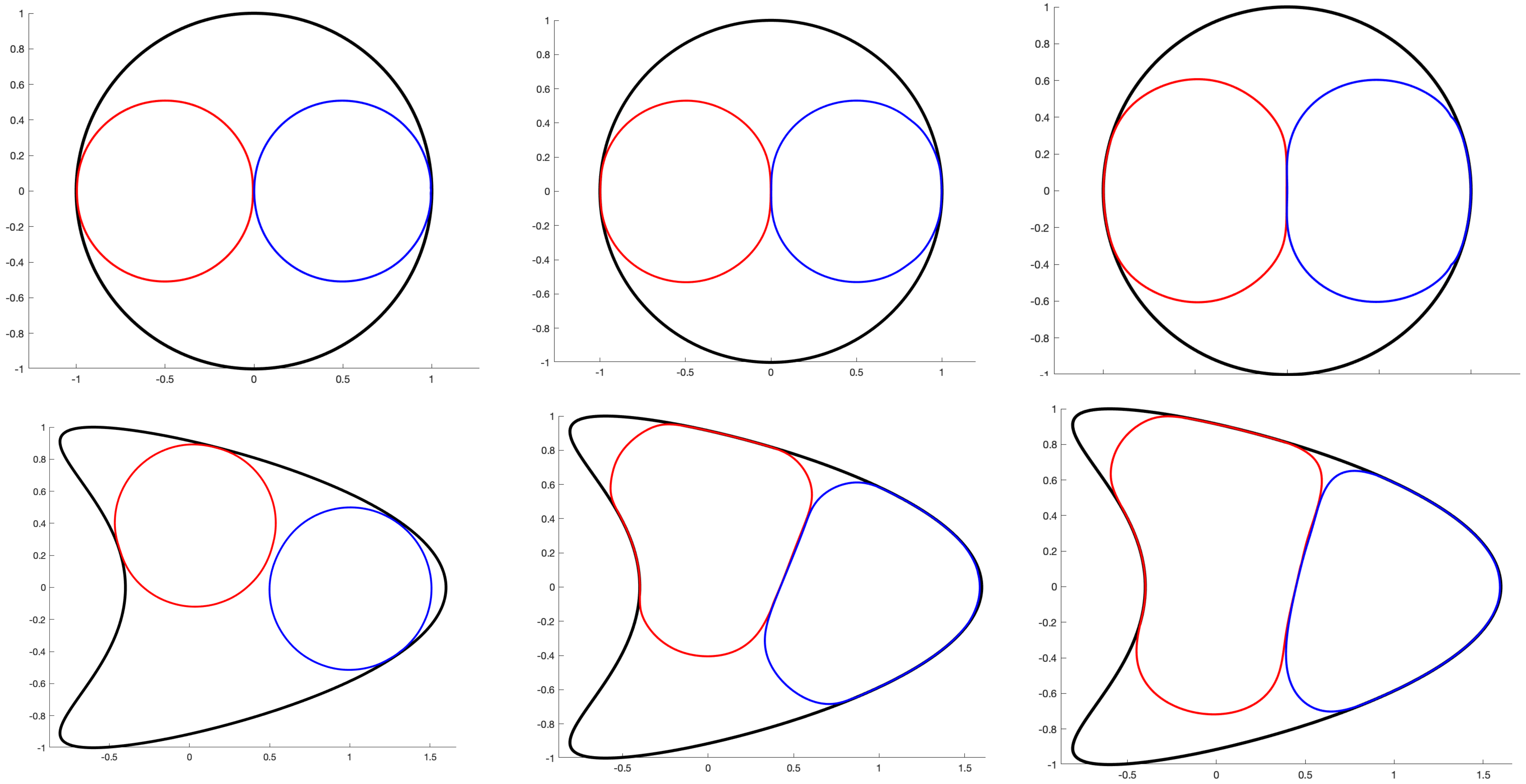
Simulations by Bourdin, Bucur, Oudet (2009)

Even more recent result: restrictions on the area

$$\min \left\{ \sum_{i=1}^{\ell} \lambda_1(\omega_i) \,\middle|\, \omega_i \subset D, \omega_i \cap \omega_j = \varnothing \;\forall\, i \neq j \text{ and } \sum_{i=1}^{\ell} \text{area}(\omega_i) \leq a \right\}$$



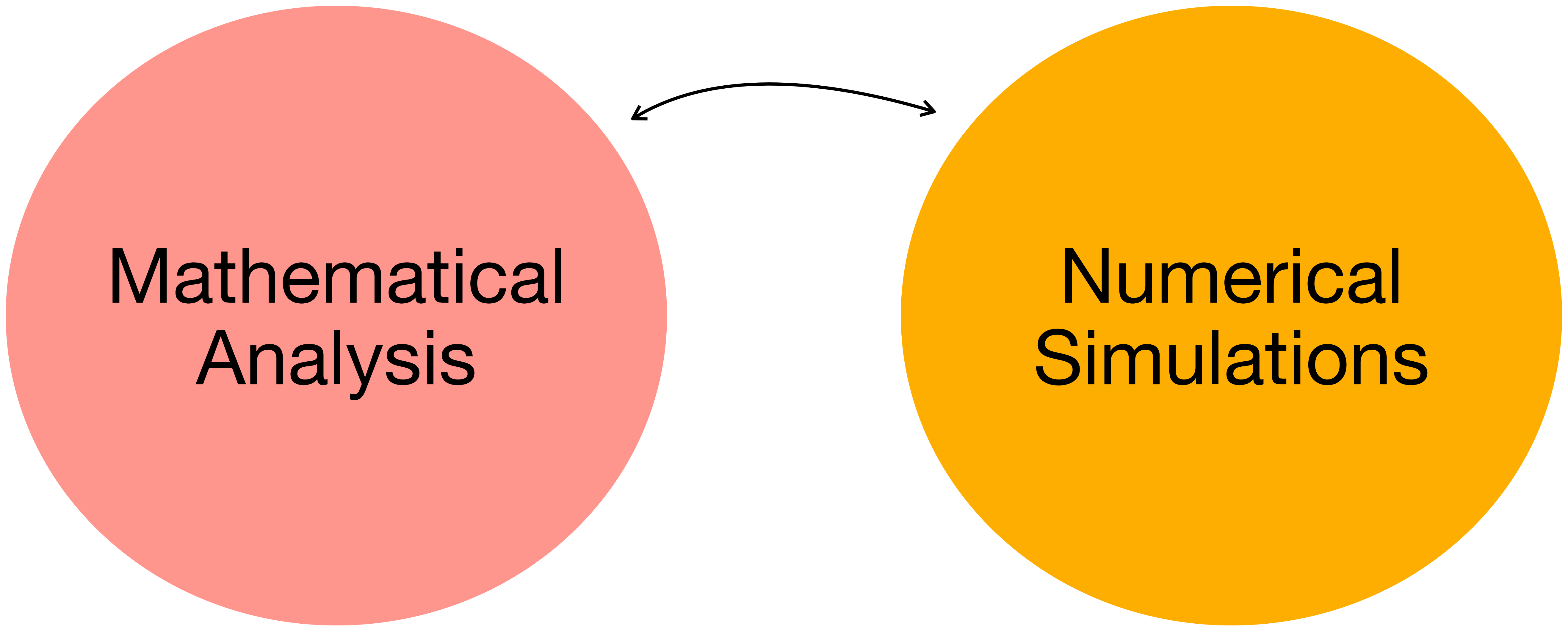
Andrade-Moreira dos Santos-Santos-Tavares 2023



Simulations by Pedro Antunes (DMIST)

Take home message

- In several contexts, it makes sense to minimize shapes
- In general: no explicit solution. One tries to obtain qualitative properties.
- Conjectures, open problems




Pedro Antunes

Do you want to know more?

https://sites.google.com/site/hugotavaresmath/

hugo.n.tavares@tecnico.ulisboa.pt