Convergence of Fourier Series and Transforms

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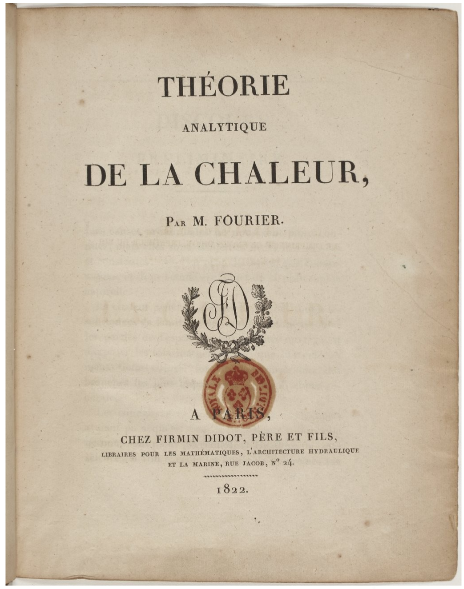

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Fourier Series

$$f : \mathbb{T} = \mathbb{R}/\mathbb{Z} \to \mathbb{C}$$

$$f \sim \sum_{j=-\infty}^{\infty} \hat{f}(j) e^{2\pi i j t}, \qquad \text{where} \qquad \hat{f}(j) = \int_{\mathbb{T}} f(t) e^{-2\pi i j t} dt$$

Fourier Transform

$$f : \mathbb{R} \to \mathbb{C}$$

$$f \sim \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i \xi t} d\xi, \qquad \text{where} \qquad \hat{f}(\xi) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i \xi t} dt$$

Some history of convergence results

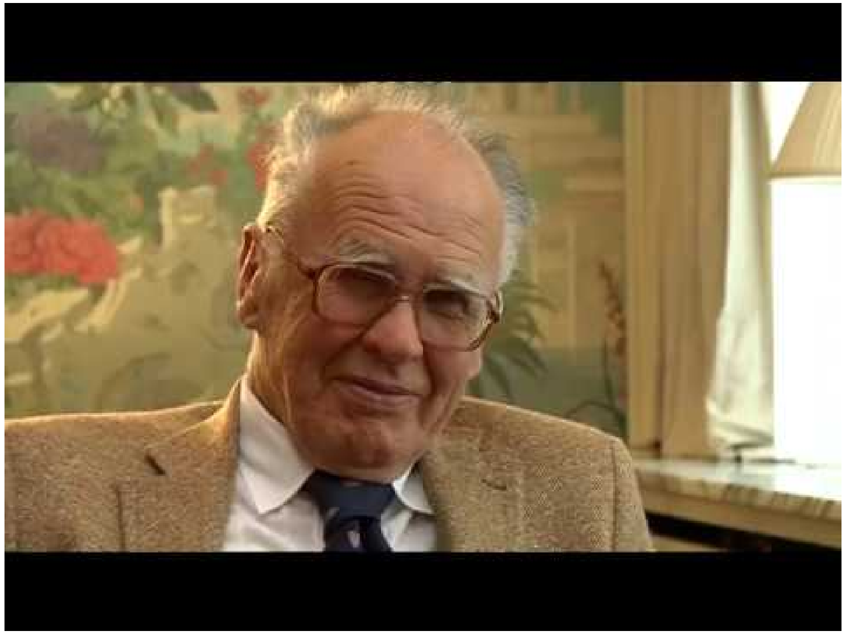
- 1829 - Dirichlet provides the first rigorous proof of convergence, for piecewise monotone functions (later extended by Jordan to BV functions)
- 1873 - du Bois-Reymond proves the existence of a continuous function whose Fourier series diverges at a point (later extended to a dense subset of continuous functions each of whose Fourier series diverges on an uncountable dense set of points)
- 1902 - Lebesgue creates the modern concept of measure and integration, which is now associated with his name, and proves that there are trigonometric series that converge everywhere to non-integrable functions.
- 1916 - Menshov constructs an example of a non-zero trigonometric series that converges to zero almost everywhere.

- 1904 - Fejér proved that, if the Fourier series of a continuous function f converges at a point x, than the limit has to be the value of the function at that point $f(x)$.
- 1907 - F. Riesz and Fischer prove that Fourier series of $L^2(\mathbb{T})$ functions converge in the $L^2(\mathbb{T})$ norm.
- 1913 - Luzin conjectures that Fourier series of $L^2(\mathbb{T})$ functions (including, in particular, continous functions) converge pointwise almost everywhere.
- 1922 - Kolmogorov (at the age of 19, before even starting his PhD) constructs an example of a function in $L^1(\mathbb{T})$ whose Fourier series diverges at every point.
- 1924 - M. Riesz proves that Fourier series of $L^p(\mathbb{T})$ functions, for $1 < p < \infty$, converge in the $L^p(\mathbb{T})$ norm.

But the problem of pointwise convergence of Fourier series for continuous functions remained completely open.

Theorem (Lennart Carleson, 1966): Fourier series of functions in $L^2(\mathbb{T})$, in particular of continuous functions, converge pointwise almost everywhere to the function.

(Extended by R. Hunt, in 1967, to any $L^p(\mathbb{T})$, $1 < p < 2$.)



- 1973 - Charles Fefferman gave a different proof of pointwise almost everywhere convergence of Fourier series for functions in $L^p(\mathbb{T})$, $p > 1$.

- 2000 - Michael Lacey and Christoph Thiele, following the ideas initially developed by Fefferman, and adapting their previous 1997 proof of the boundedness of the bilinear Hilbert transform, provided what is currently considered the modern proof that Fourier series converge almost everywhere, for $f \in L^p(\mathbb{T})$, $p > 1$, using time-frequency dyadic methods.

The Goal

To prove the weak L^2 boundedness of the Carleson operator

$$\left\| \sup_N \left| \sum_{j=-N}^{N} \hat{f}(j)\, e^{2\pi i j t} \right| \right\|_{L^{2,\infty}(\mathbb{T})} \le C \|f\|_{L^2(\mathbb{T})}$$