The mathematics of waves and black holes

Master in Applied Mathematics and Computation

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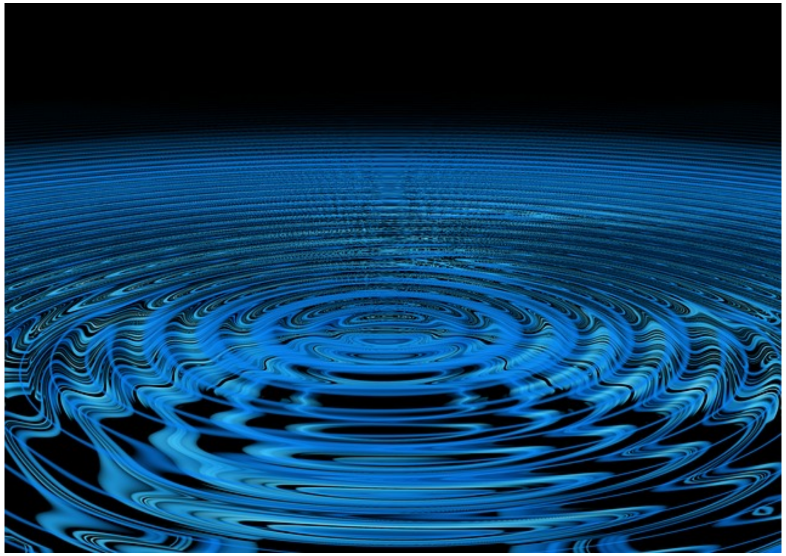
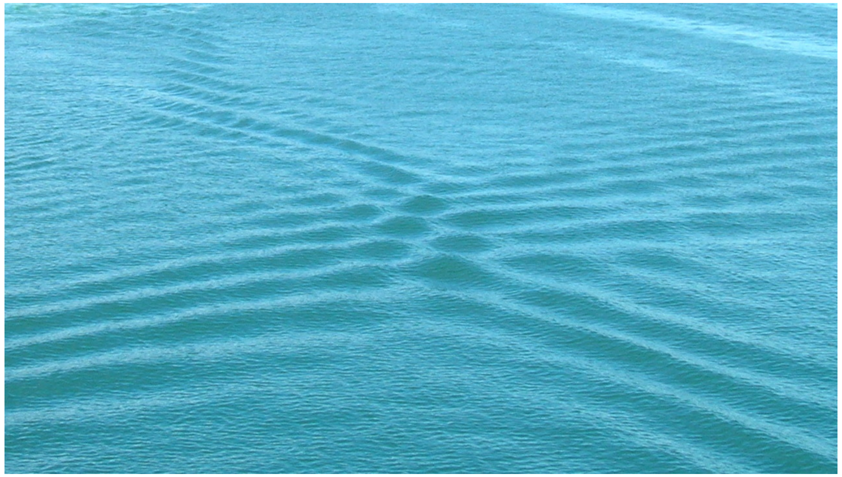
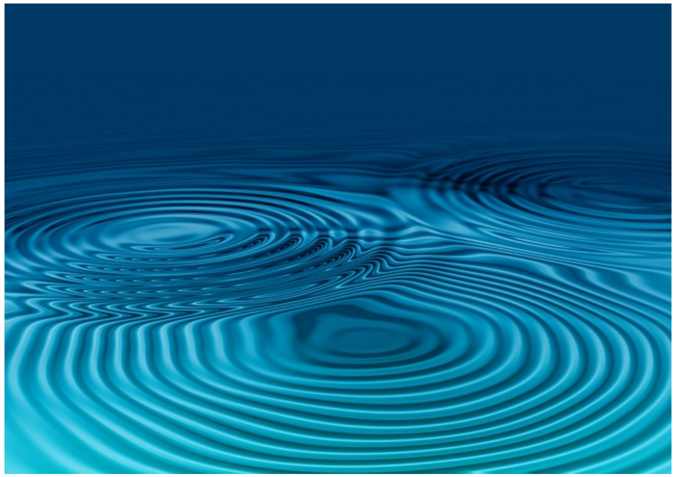
Waves

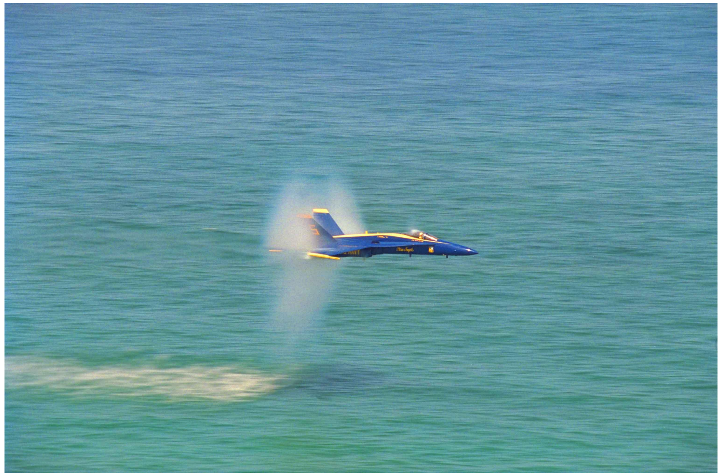


Linear waves

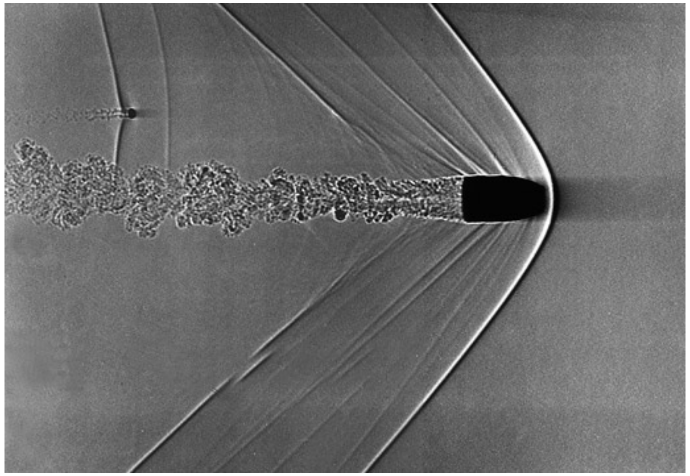


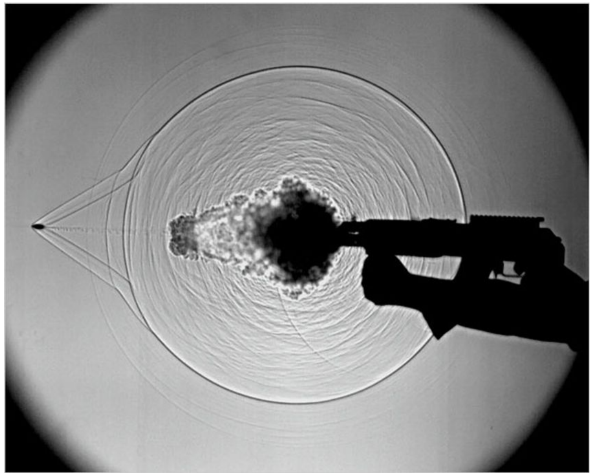


Sound waves



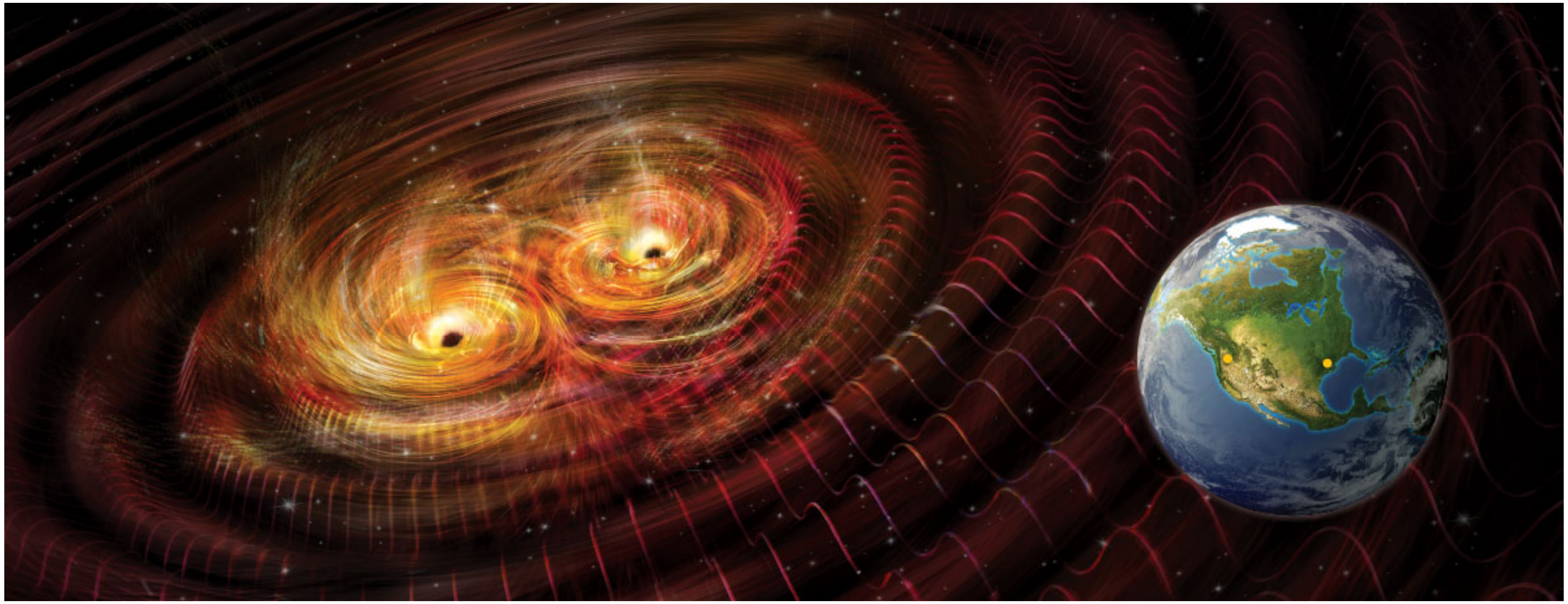
(NavSource Naval History)





Electromagnetic waves

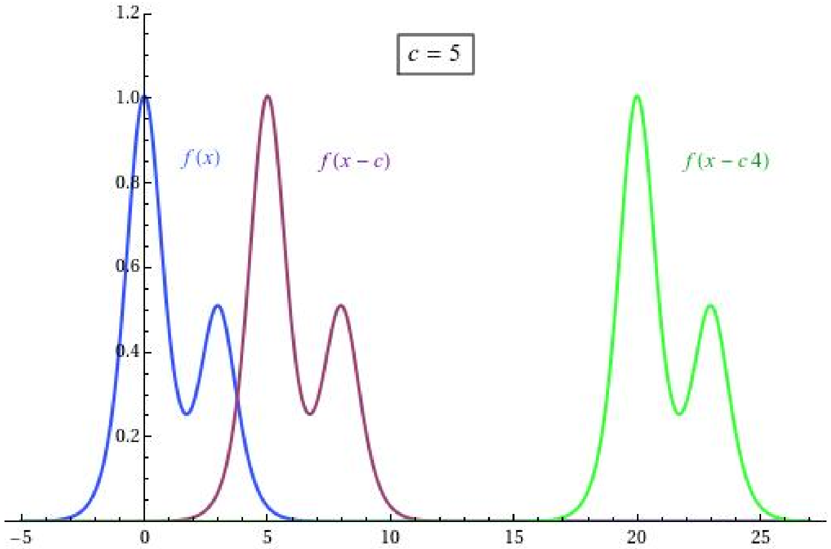
Gravitational waves

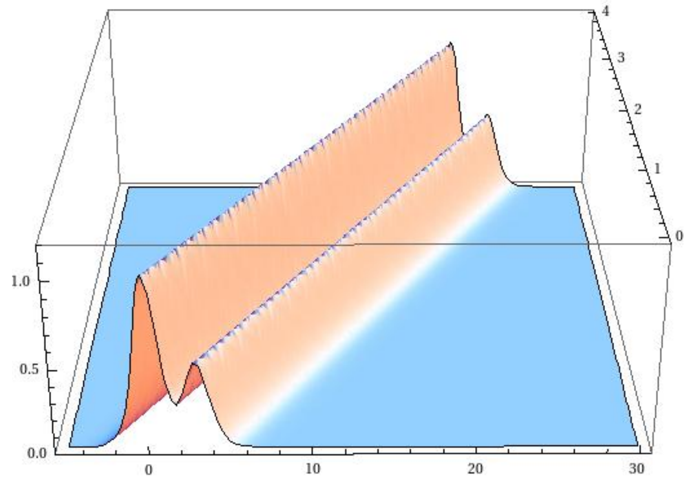


Mathematical definition of a wave

Prototype: traveling wave with velocity c.

$$u(x,t) = f(x - ct), \qquad c \in \mathbb{R}$$





Wave equation in one dimension

D'Alembert (1747) deduced the one-dimensional wave equation to study the vibration of a string:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad \left(c^2 = \frac{T}{\mu}\right)$$

Changing variables to $\xi = x - ct$, $\eta = x + ct$, the wave equation is written

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0,$$

and can be solved by integrating twice:

$$u = f(\xi) + g(\eta) = f(x - ct) + g(x + ct)$$

Wave equation in three dimensions

The generalization to three dimensions is obvious:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

This is a linear partial differential equation, and gives a good description of (linear) sound waves, electromagnetic waves, and (linear) gravitational waves.

It admits a conserved energy:

$$E(t) = \int_{\mathbb{R}^3} \left[\frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right] dxdydz$$

If u is a solution of the wave equation, then so are $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$. Therefore we also have the conserved generalized energies

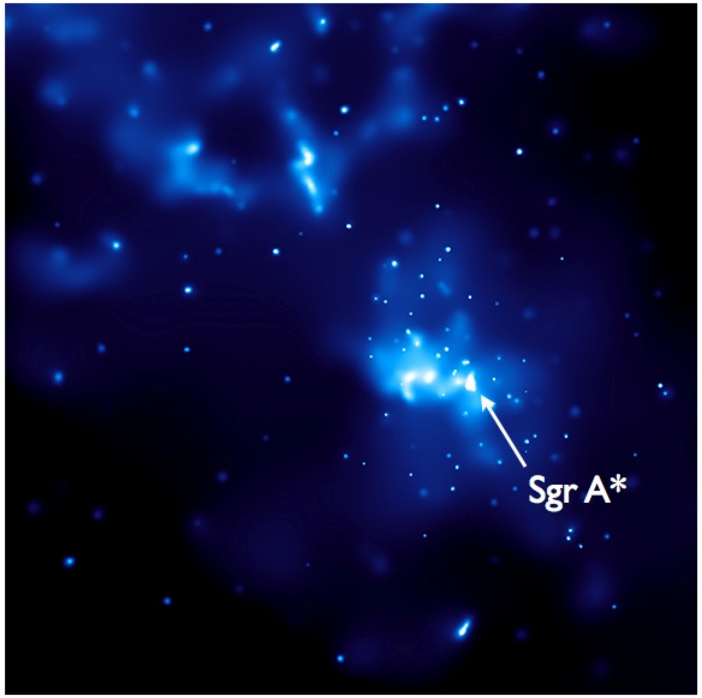
$$E_x(t) = \int_{\mathbb{R}^3} \left[\frac{1}{c^2} \left(\frac{\partial^2 u}{\partial t \partial x} \right)^2 + \left(\frac{\partial^2 u}{\partial x^2} \right)^2 + \left(\frac{\partial^2 u}{\partial y \partial x} \right)^2 + \left(\frac{\partial^2 u}{\partial z \partial x} \right)^2 \right] dxdydz$$

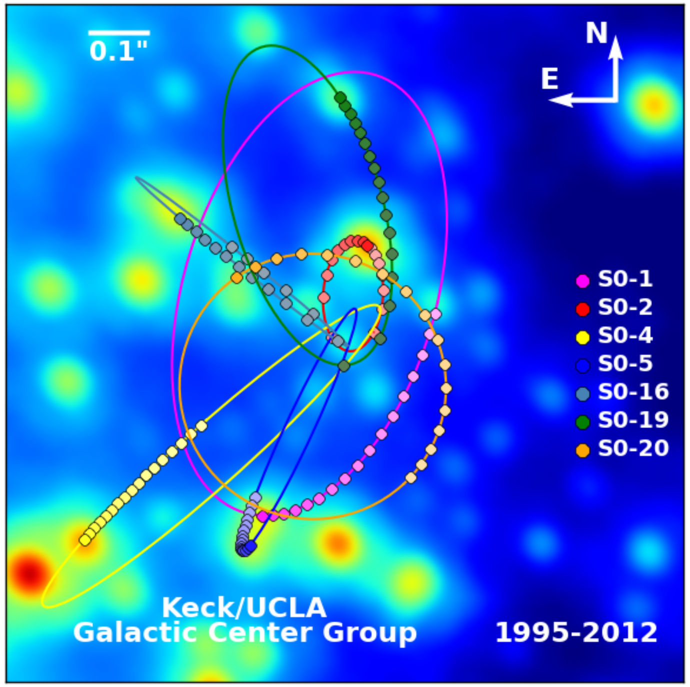
$$E_y(t) = \int_{\mathbb{R}^3} \left[\frac{1}{c^2} \left(\frac{\partial^2 u}{\partial t \partial y} \right)^2 + \left(\frac{\partial^2 u}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 u}{\partial y^2} \right)^2 + \left(\frac{\partial^2 u}{\partial z \partial y} \right)^2 \right] dxdydz$$

$$E_z(t) = \int_{\mathbb{R}^3} \left[\frac{1}{c^2} \left(\frac{\partial^2 u}{\partial t \partial z} \right)^2 + \left(\frac{\partial^2 u}{\partial x \partial z} \right)^2 + \left(\frac{\partial^2 u}{\partial y \partial z} \right)^2 + \left(\frac{\partial^2 u}{\partial z 2} \right)^2 \right] dxdydz$$

These can be used together with Sobolev's inequality to show that solutions with finite generalized energies are bounded (and even that they decay in time).

Black holes


Sgr A*


0.1"
N
E
S0-1
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Keck/UCLA
Galactic Center Group
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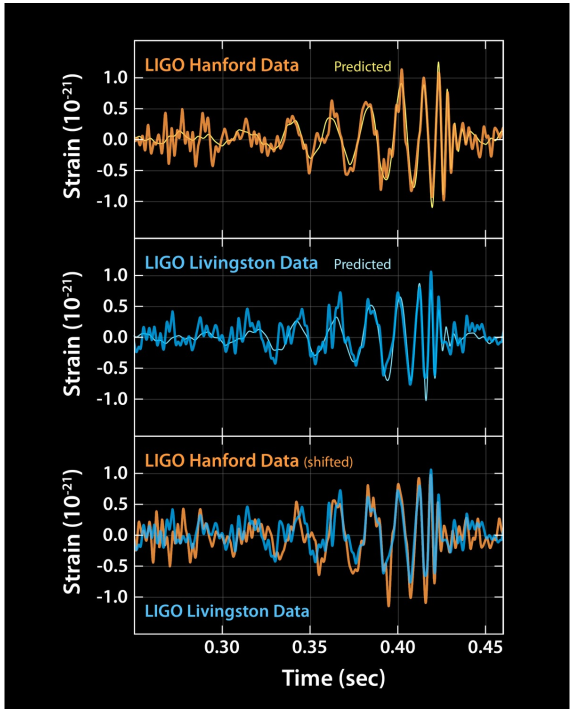
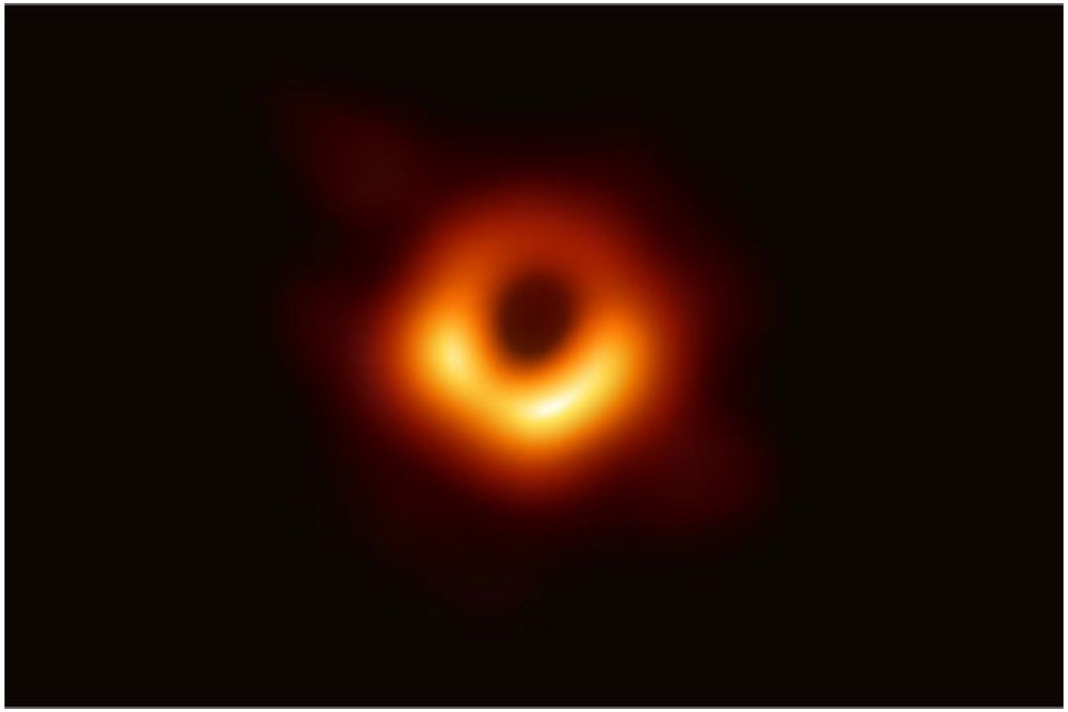
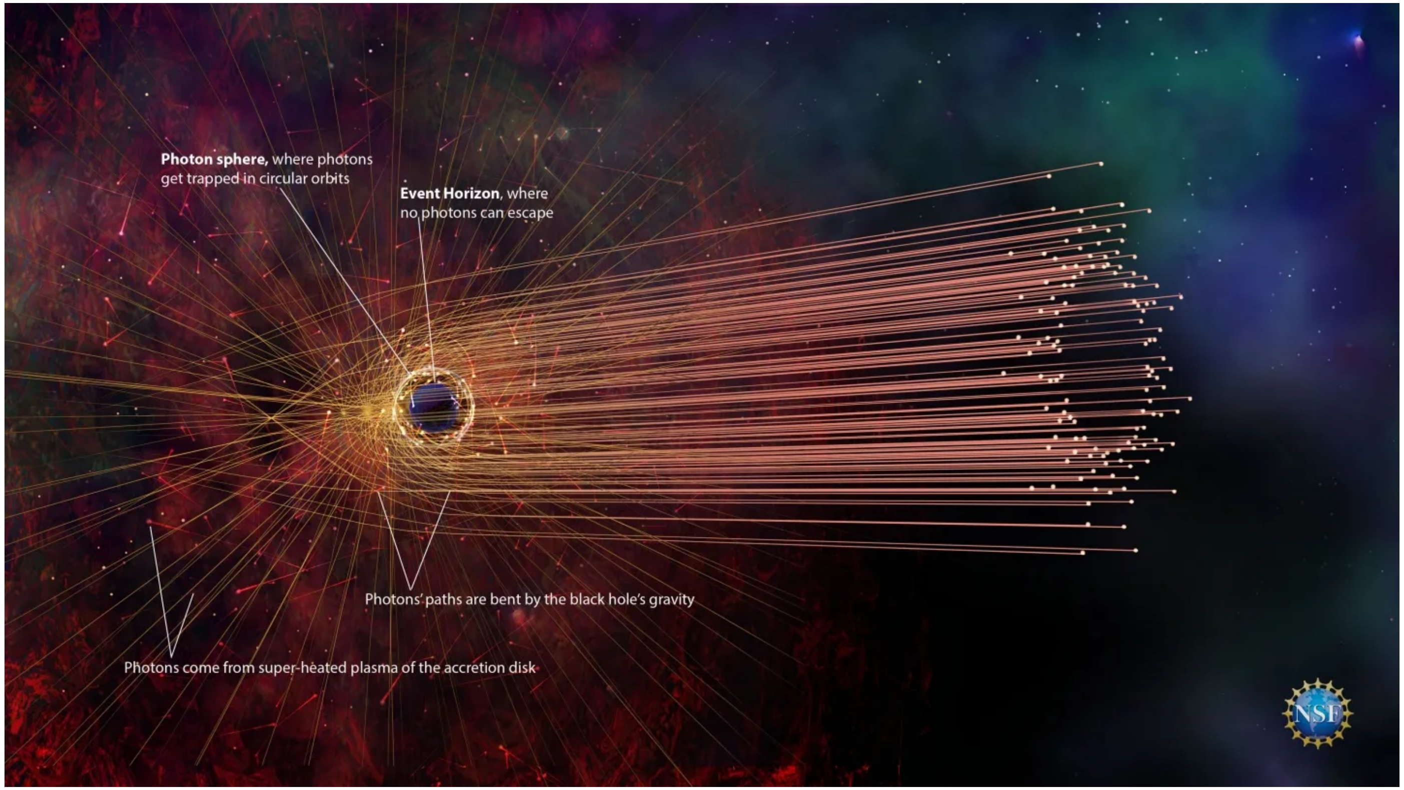

LIGO Hanford Data
Predicted
LIGO Livingston Data
Predicted
LIGO Hanford Data (shifted)
LIGO Livingston Data
Strain (10^-21)
Strain (10^-21)
Strain (10^-21)
1.0
0.5
0.0
-0.5
-1.0
0.30
0.35
0.40
0.45
Time (sec)



Photonsphere



Wave equation on a black hole background

The spacetime around a spherical black hole of mass M is modeled by the Schwarzschild Lorentzian metric:

$$g = -V dt \otimes dt + V^{-1} dr \otimes dr + r^2 \left(d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi \right)$$

where

$$V = 1 - \frac{2M}{r}$$

One can still write the wave equation by using the Levi-Civita connection of this metric:

$$d \star du = 0 \Leftrightarrow \nabla^\alpha \nabla_\alpha u = 0$$

If $M = 0$ this is the usual wave equation written in spherical coordinates.

One can use the divergence theorem to prove that again there is a conserved energy:

$$E(t) = \int_{\Sigma_t} T\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)\left(1 - \frac{2M}{r}\right)^{-1} r^2 \sin\theta dr d\theta d\varphi$$

where

$$T = du \otimes du - \frac{1}{2} g^{-1}(du, du)\, g$$

is the energy-momentum tensor associated to u. Again using the Sobolev inequality, one can prove boundedness (and even decay) of finite energy solutions. New challenges: the event horizon and photonsphere trapping.

Are they stable?