Smooth symmetries of spheres

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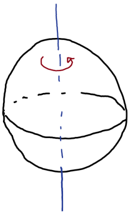
Isometries of the sphere

n-sphere $S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$

Definition

An *isometry* is a bijection $f : S^n \to S^n$ which preserves distances.

$n = 1$: rotation by an angle θ (and reflection)
$n = 2$: rotation about an axis in $\mathbb{R}^3$ by an angle θ (and reflection).



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Any isometry $f : S^n \to S^n$ can be extended to a linear isometry

$$F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \quad , \quad F(x) = f(x/||x||)||x||$$

i.e. an orthogonal transformation. That is,

$$\mathrm{Isom}(S^n) \cong O(n+1)$$

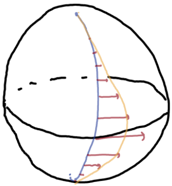
Smooth symmetries of the sphere

isometries $\subset$ smooth symmetries

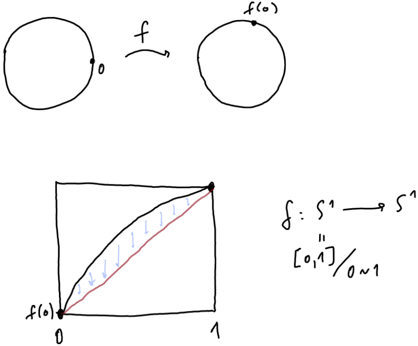
A smooth symmetry – aka *diffeomorphism* – is a smooth map $S^n \to S^n$ with smooth inverse.

$$O(n+1) \subset \mathrm{Diff}(S^n)$$

$\rightsquigarrow$ "Most" smooth symmetries aren't isometries.



Smooth symmetries of the circle



Therefore, $\mathrm{Diff}(S^1) \simeq O(2)$, i.e. the inclusion has an inverse up to deformation (= homotopy equivalence).

Smooth symmetries $\simeq$ isometries?

$\mathrm{Diff}(S^1) \simeq O(2)$
$\mathrm{Diff}(S^2) \simeq O(3)$ (Smale '50s)

DIFFEOMORPHISMS OF THE 2-SPHERE

STEPHEN SMALE[1]

The object of this paper is to prove the theorem.

THEOREM A. *The space Ω of all orientation preserving C^∞ diffeomorphisms of S^2 has as a strong deformation retract the rotation group $SO(3)$.*

Here S^2 is the unit sphere in Euclidean 3-space, the topology on Ω is the C^r topology $\infty \geqq r > 1$ (see [4]) and a diffeomorphism is a differentiable homeomorphism with differentiable inverse.

Smooth symmetries $\simeq$ isometries?

$\mathrm{Diff}(S^1) \simeq O(2)$
$\mathrm{Diff}(S^2) \simeq O(3)$ (Smale '50s)
$\mathrm{Diff}(S^3) \simeq O(4)$ (Hatcher '80s)

Annals of Mathematics, 117 (1983), 553–607

A proof of the Smale Conjecture, $\mathrm{Diff}(S^3) \simeq O(4)$

By Allen E. Hatcher

The Smale Conjecture [9] is the assertion that the inclusion of the orthogonal group $O(4)$ into $\mathrm{Diff}(S^3)$, the diffeomorphism group of the 3-sphere with the C^∞ topology, is a homotopy equivalence. There are many equivalent forms of this

$\mathrm{Diff}(S^4) \not\simeq O(5)$ (Watanabe 2018, arXiv)

SOME EXOTIC NONTRIVIAL ELEMENTS OF THE RATIONAL HOMOTOPY GROUPS OF $\mathrm{Diff}(S^4)$

TADAYUKI WATANABE

ABSTRACT. This paper studies the rational homotopy groups of the group $\mathrm{Diff}(S^4)$ of self-diffeomorphisms of S^4 with the C^∞-topology. We present a method to prove that there are many 'exotic' non-trivial elements in $\pi_*\mathrm{Diff}(S^4)\otimes\mathbb{Q}$ parametrized by trivalent graphs. As a corollary of the main result, the 4-dimensional Smale conjecture is disproved. The proof utilizes Kontsevich's characteristic classes for smooth disk bundles and a version of clasper surgery for families. In fact, these are analogues of Chern–Simons perturbation theory in 3-dimension and clasper theory due to Goussarov and Habiro.

1. Introduction

The homotopy type of $\mathrm{Diff}(S^4)$ is an important object in topology, whereas almost nothing was known about its homotopy groups except that they include those coming from the orthogonal group O_5 (e.g., recent surveys in [Hat2, Kup]). Let $\mathrm{Diff}(D^d, \partial)$ denote the group of self-diffeomorphisms of D^d which fix a neighbor-

Exotic smooth bundles

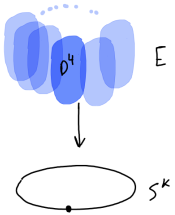
$$\mathrm{Diff}(S^n) \simeq O(n+1) \times \mathrm{Diff}_*(S^n)$$

So Watanabe's result is equivalent to

$$\mathrm{Diff}_*(S^4) \not\simeq *$$

Strategy: Test with spheres,

$\{S^{k-1} \to \mathrm{Diff}_*(S^4)\} \leftrightarrows \{$smooth S^k-families of (pointed) 4-spheres$\}$



If $\mathrm{Diff}_*(S^4) \simeq *$, then every smooth S^k-family of 4-spheres is trivial, i.e. a product $S^k \times S^4$

Configuration space integrals

Watanabe constructs exotic S^k families of 4-spheres (many! for $k = 2, 5, 9, \ldots$).To check non-triviality, he uses an algebraic invariant. For each trivalent graph Γ with $2k$-vertices:

$$\{\text{smooth } S^k\text{-families of (pointed) 4-spheres}\} \to \mathbb{R}$$

$$(S^4 \to E \to S^k) \mapsto \int_{C_k(E)} \omega_\Gamma$$

Many questions ... e.g. does $\int$ depend on the smooth structure of the fibers? (Lin-Xie '23) It depends on less. But word is: it may depend on even less.