Space-Time Extremes
&
Applications to Environmental Data

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Outline

- Extremes and Maximum Domain of Attraction (MDA)
- Extreme value index - $\gamma \in \mathbb{R}$ -
- Extensions of MDA to more dimensions
- Applications

Probability and Statistics

Extreme Values commonly relate to maxima and minima,

$$X_{n,n} = \max(X_1, X_2, \dots, X_n) \qquad X_{1,n} = \min(X_1, X_2, \dots, X_n)$$

with $X_1, \dots, X_n$ a sample of independent and identically distributed (common d.f. F) random variables.

Let's look at the d.f. of the maxima:

$$\max(X_1, X_2, \dots, X_n) \le x \qquad \text{iif} \qquad X_1 \le x,\ X_2 \le x,\ \dots,\ X_n \le x$$

hence,

$$P(X_{n,n} \le x) = \prod_{i=1}^{n} P(X_i \le x) = F^n(x), \qquad x \in \mathbb{R}.$$

Maximum Domain of Attraction condition and GEV$_\gamma$ d.f.s

If $\exists\ a_n > 0$ and $b_n \in \mathbb{R}$ s.t. convergence in distribution holds,

$$\lim_{n\to\infty} P\left(\frac{X_{n,n} - b_n}{a_n} \le x\right) = \lim_{n\to\infty} F^n\left(a_n x + b_n\right) = G(x) \quad \text{exists,}$$

then

$$G_\gamma(x) = e^{-(1+\gamma x)^{-1/\gamma}}, \quad 1 + \gamma x > 0,\ \gamma \in \mathbb{R}.$$

Central Limit Theorem and Gaussian distribution:

with $E[X_i] = \mu$, $var(X_i) = \sigma^2 < \infty$, convergence in distribution holds,

$$\frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n\sigma^2}} = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim^a N(0,1)$$

$$\lim_{n\to\infty} P\left(\frac{\sum_{i=1}^n X_i - n\mu}{\sigma\sqrt{n}} \le z\right) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx.$$

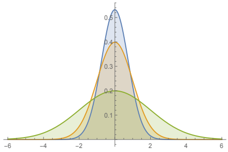


Figure: $\mathrm{N}(\mu, \sigma^2)$

MDA and the extreme value index - $\gamma \in \mathbb{R}$ -

Examples:

1. - $\gamma = 1$ - Pareto $F_X(x) = 1 - 1/x$, $x > 1$:

$$F^n(a_n x + b_n) = \left(1 - \frac{1}{a_n x + b_n}\right)^n_{\Big| a_n = n, b_n = n}$$

$$= \left(1 - \frac{(x+1)^{-1}}{n}\right)^n \to_{n\to\infty} e^{-(x+1)^{-1}}, \; x + 1 > 0.$$

2. - $\gamma = 0$ - Exponential $F_X(x) = 1 - e^{-x}$, $x > 0$:

$$F^n(a_n x + b_n) = \left(1 - e^{-a_n x - b_n}\right)^n_{\Big| a_n = 1, b_n = \log n}$$

$$= \left(1 - \frac{e^{-x}}{n}\right)^n \to_{n\to\infty} e^{-e^{-x}}, \; x > 0.$$

Extreme value index - $\gamma \in \mathbb{R}$ -

- $\gamma > 0$: $1 - G_\gamma(x) \sim (\gamma \mathbf{x})^{-\mathbf{1}/\gamma}, \quad x \to \infty$
- $\gamma = 0$: $1 - G_0(x) \sim \mathbf{e}^{-\mathbf{x}}, \quad x \to \infty$
- $\gamma < 0$: $1 - G_\gamma\left(-\gamma^{-1} - x\right) \sim (-\gamma \mathbf{x})^{-\mathbf{1}/\gamma}, \quad x \downarrow 0.$

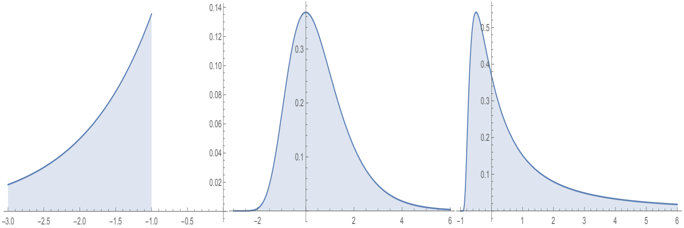


Figure: GEV densities: $\gamma = -1, 0, 1$.

Extreme value index estimation

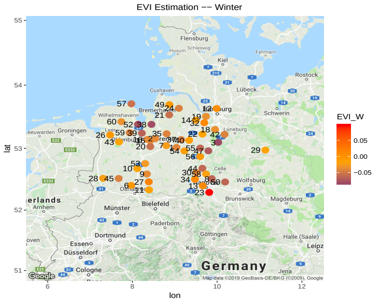


Figure: Estimation of the extreme value index for precipitation data from observational stations

- extreme value index, j-th moment of log-spacings

$$\sum_{i=0}^{k-1} (\log X_{n-i,n} - \log X_{n-k,n})^j$$

- exceedance probabilities
- high quantiles
- value-at-risk
- return value
- return period

Spatial-temporal data

Observed space-time process $X = \{X_{i,j}\}_{i,j}^{n,m}$

- (time - days $i = 1, \ldots, n$)
- (space - stations $j = 1, \ldots, m$)

$(X_{i,1}, X_{i,2}, \ldots, X_{i,m})$ joint dependence $F_{i;1,\ldots,m}$

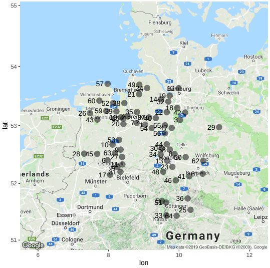


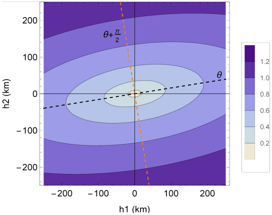
Figure: Weather stations in NW Germany

Bivariate Maximum Domain of Attraction condition

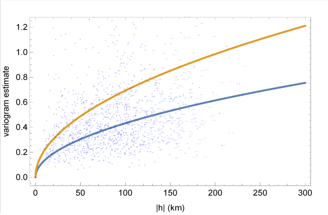
$\mathrm{GEV}_{\gamma_1,\gamma_2}$ d.f.s: $\exists\ a_n,\ c_n > 0,\ b_n\, d_n \in \mathbb{R}$ s.t. conv. in distrib. holds,

$$\lim_{n\to\infty} P\left(\frac{X_{n,n} - b_n}{a_n} \le x,\ \frac{Y_{n,n} - d_n}{c_n} \le y\right)$$
$$= \lim_{n\to\infty} F^n\left(a_n x + b_n,\ c_n y + d_n\right) = G_{\gamma_1,\gamma_2}(x,y),\ \gamma_1,\gamma_2 \in \mathbb{R}.$$

Winter variogram and non-stationary stations



(a)



(b)

Figures (a)&(b). Winter variogram

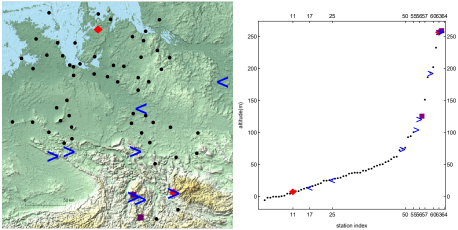


Figure 2.

Non-stationary

stations over time

and over space

THANK YOU !!