Adaptive Finite Element Methods for Contact Problems

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Contact problem simulations

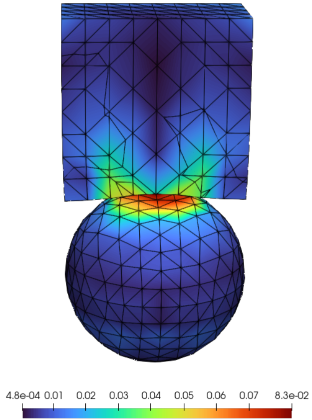


Figure: Von Mises stress in a 3D contact between two elastic bodies.

Contact problem simulations

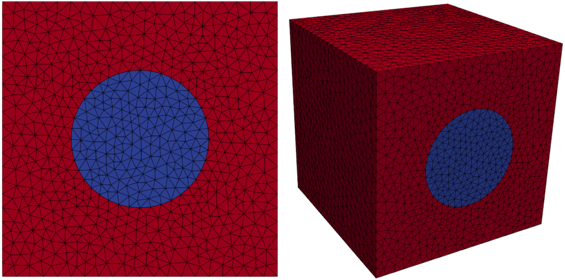


Figure: Cylindrical shaft cutting through a 3D block.

Contact problem simulations

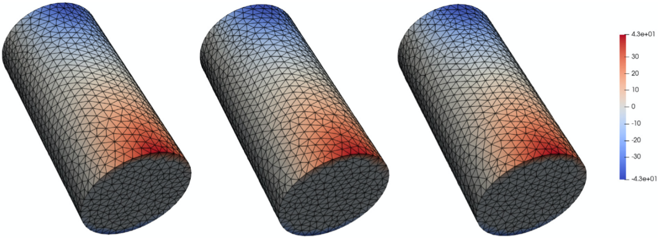


Figure: The normal contact loads for three different angles.

Contact problem simulations

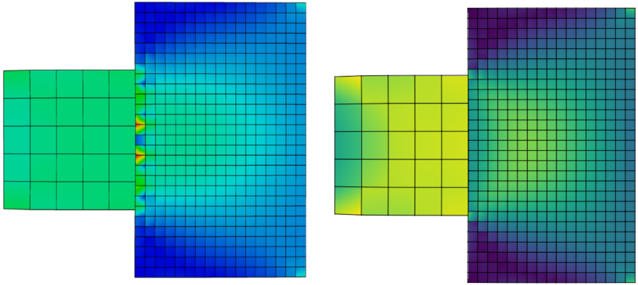


Figure: Most commercial finite element codes implement contact constraints node-wise. These so-called node-to-segment strategies can cause unphysical oscillations when the vertices of the two meshes do not match. The numerical solution on the left was computed using a recent version of the finite element solver `Abaqus` and the solution on the right was computed using Nitsche's method.

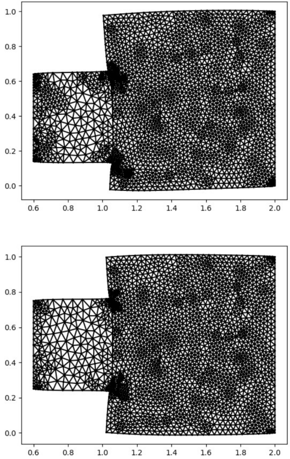
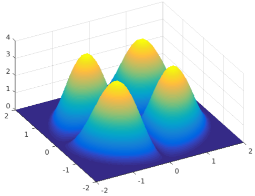


Figure: Adaptive meshes are needed for the contact between two elastic bodies.

Obstacle problem



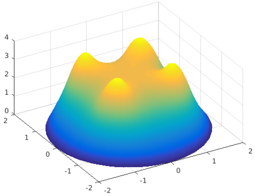


Figure: The obstacle g (left) and the solution u (right).

Obstacle problem

$$\begin{aligned} -\Delta u &\geq f \quad \text{in } \Omega \subset \mathbb{R}^2, \\ u &\geq g \quad \text{in } \Omega, \\ (u-g)(\Delta u + f) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Above, the unknown $u : \Omega \to \mathbb{R}$ describes the vertical displacement of a membrane, constrained to lie above a rigid body, under a body loading f.

Lagrange multiplier formulation

$$
\begin{aligned}
-\Delta u - \lambda = f \quad & \text{in } \Omega, \\
u - g \geq 0 \quad & \text{in } \Omega, \\
\lambda \geq 0 \quad & \text{in } \Omega, \\
(u - g)\lambda = 0 \quad & \text{in } \Omega, \\
u = 0 \quad & \text{on } \partial\Omega.
\end{aligned}
$$

The new variable $\lambda = -\Delta u - f$ is a Lagrange multiplier imposing the constraint $u \geq g$ and can be interpreted as the contact force between the membrane and the obstacle.

Variational formulation

Let V and Q be some suitable function spaces and define

$$\Lambda = \{\mu \in Q : \langle v, \mu \rangle \geq 0 \ \forall v \in V \text{ with } v \geq 0 \text{ in } \Omega\} .$$

Variational problem: Find $(u, \lambda) \in V \times \Lambda$ such that

$$\begin{aligned} (\nabla u, \nabla v) - \langle v, \lambda \rangle = (f, v) \quad & \forall v \in V, \\ \langle u - g, \mu - \lambda \rangle \geq 0 \quad & \forall \mu \in \Lambda . \end{aligned}$$

Minimization problem

The obstacle problem is an archetypical example of a constrained minimization problem (CMP), in this case of the form

$$\min_{\substack{u \in V \\ u \geq g}} \; \frac{1}{2} (\nabla u, \nabla u) - (f, u) \, .$$

CMPs are often approximated by mixed Finite Element methods based on the Lagrange multiplier formulation.

Finite Element Approximation

Let $\mathcal{C}_h$ denote a finite element mesh consisting of triangular elements K and let $V_h \subset V$ and $Q_h \subset Q$ be the finite element spaces

$$V_h = \{v_h \in V : v_h|_K \in P_1(K)\ \forall K \in \mathcal{C}_h\}$$
$$Q_h = \{\xi_h \in Q : \xi_h|_K \in P_0(K)\ \forall K \in \mathcal{C}_h\}$$

where $P_k(K)$ denotes the space of polynomials of degree k in K.

Moreover, let

$$\Lambda_h = \{\mu_h \in Q_h : \mu_h \geq 0 \text{ in } \Omega\} \subset \Lambda,$$

After calculating the discrete solution corresponding to the mesh $\mathcal{C}_h$, we evaluate an error indicator η_K at each element $K \in \mathcal{C}_h$. The error indicator depends only on the discrete solution and measures how well the discrete solution satisfies the continuous problem at each element.

Next, we decide which elements to refine based on the values of the error indicators.

It holds that

A Posteriori Estimate

$$\|u - u_h\| + \|\lambda - \lambda_h\| \lesssim \sum_{K \in \mathcal{C}_h} \eta_K \lesssim \|u - u_h\| + \|\lambda - \lambda_h\|$$

Adaptive FEM

Adaptive finite element method thus requires

- an initial mesh;
- an error indicator computed at each element from the discrete solution;
- a marking strategy which, given the values of the error indicators, decides where to refine;
- a mesh refinement algorithm producing the next mesh.

Adaptive schemes need less degrees-of-freedom than uniform ones to achieve a given accuracy and whereas the convergence of the uniform strategy is limited by the regularity of the exact solution, the adaptive strategy regains the optimal convergence rate.

At an implementational level, the Lagrange multiplier variable can be eliminated and the method reads simply as follows

$$(\nabla u_h, \nabla v_h) + (\hbar^{-2} u_h, v_h)_{\Omega_h^c} = (f, v_h)_{\Omega \setminus \Omega_h^c} + (\hbar^{-2} g, v_h)_{\Omega_h^c}$$

where $\Omega_h^c = \{\, (x,y) \in \Omega \,|\, \lambda_h(x,y) > 0 \,\}$ is the unknown contact region.

Inequality-constrained problems are nonlinear due to the unknown contact region Ω_h^c. In practice, one can find the set Ω_h^c efficiently using an iterative Newton-type linearisation algorithm.

Proposal for a MSc thesis

MSc thesis proposal:

- derivation of adaptive strategies (error indicators) for obstacle-type problems
- implementation of adaptive FE methods for these problems using, for example, `scikit-fem`:

  `https://scikit-fem.readthedocs.io/en/latest/`

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