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Artificial boundary conditions for fluid flow simulation

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Fluids are everywhere

Fluid mechanics is a part of applied mathematics, of physics, of many branches of engineering, certainly civil, mechanical, chemical, and aeronautical engineering, and of naval architecture and geophysics, with astrophysics and biological and physiological fluid dynamics to be added.

Sydney Goldstein, mathematician and fluid-dynamicist

Incompressible Navier-Stokes Equations (NSE)

$$\varrho\,[\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}] = \mu\Delta\boldsymbol{v} - \nabla p + \varrho\boldsymbol{f}, \quad \nabla\cdot\boldsymbol{v} = 0$$

Although the NSE are known for almost two centuries and the wide range of their scientific and technological applications, the theory of NSE is far from complete.

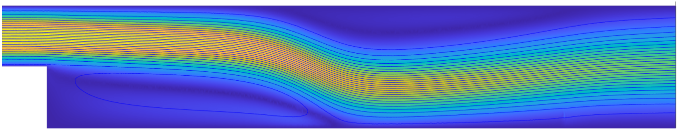
It is known that a 2D weak solution exists and is unique.

The question whether a 3D weak solution, which exists globally in time, is smooth for all times without restrictions on the size of the initial velocity and the external force, has challenged several generations of mathematicians and is one of the seven *Millennium Prize Problems* selected by the Clay Mathematics Institute.

Artificial boundary conditions for fluid flow simulation

Flow in pipes

Navier-Stokes equations with Classical Do-Nothing (CDN) boundary condition



$$\begin{cases} -\nu\Delta\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p = \boldsymbol{f} \text{ in } \Omega \subset \mathbb{R}^n,\ n \in \{2,3\} \\ \nabla \cdot \boldsymbol{v} = 0 \text{ in } \Omega \\ \boldsymbol{v} = \boldsymbol{g} \text{ on } \Gamma_{\mathrm{D}} \text{ (\textbf{Dirichlet} boundary portion)} \\ \nu(\boldsymbol{n}\cdot\nabla)\boldsymbol{v} - p\boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{N}} \text{ (\textbf{Neumann} boundary portion)} \end{cases}$$

Γ_{N} is an artificial boundary

Navier-Stokes equations with CDN boundary condition

Pseudostress tensor:

$$\widetilde{\mathbb{T}}(\boldsymbol{v}, p) := \nu \nabla \boldsymbol{v} - p\mathbb{I}, \quad \nu \text{ viscosity constant}$$

$$(\text{NS+CDN}) \begin{cases} -\nabla \cdot \widetilde{\mathbb{T}}(\boldsymbol{v}, p) + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \boldsymbol{f} \text{ in } \Omega \\ \nabla \cdot \boldsymbol{v} = 0 \text{ in } \Omega \\ \boldsymbol{v} = \boldsymbol{g} \text{ on } \Gamma_{\mathsf{D}} \\ \boldsymbol{n} \cdot \widetilde{\mathbb{T}}(\boldsymbol{v}, p) = \mathbf{0} \text{ on } \Gamma_{\mathsf{N}} \text{ (compatible with Poiseuille flow)} \end{cases}$$

Directional Do-Nothing (DDN) boundary condition

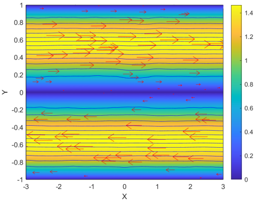
Energy estimate: if $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$ with $\nabla \cdot \boldsymbol{v} = 0$ and $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$ with $\boldsymbol{u} = \mathbf{0}$ on Γ_D then

$$\int_\Omega \boldsymbol{v} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u} dx = \frac{1}{2} \int_{\Gamma_\mathsf{N}} (\boldsymbol{v} \cdot \boldsymbol{n}) |\boldsymbol{u}|^2 d\sigma$$

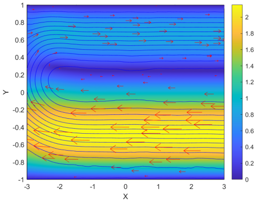
$$= \frac{1}{2} \int_{\Gamma_\mathsf{N}} [\boldsymbol{v} \cdot \boldsymbol{n}]^+ |\boldsymbol{u}|^2 d\sigma - \frac{1}{2} \int_{\Gamma_\mathsf{N}} [\boldsymbol{v} \cdot \boldsymbol{n}]^- |\boldsymbol{u}|^2 d\sigma$$

$$\text{(NS+DDN)} \quad \begin{cases} -\nabla \cdot \widetilde{\mathbb{T}}(\boldsymbol{v}, p) + \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{v}) = \boldsymbol{f} \text{ in } \Omega \\ \nabla \cdot \boldsymbol{v} = \mathbf{0} \text{ in } \Omega \\ \boldsymbol{v} = \boldsymbol{g} \text{ on } \Gamma_\mathsf{D} \\ \boldsymbol{n} \cdot \widetilde{\mathbb{T}}(\boldsymbol{v}, p) + \frac{1}{2} [\boldsymbol{v} \cdot \boldsymbol{n}]^- (\boldsymbol{v} - \boldsymbol{v}_r) = \mathbf{0} \text{ on } \Gamma_\mathsf{N} \end{cases}$$

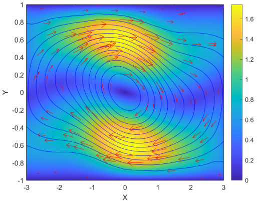
CDN vs DDN boundary condition: truncated pipe



(a) CDN-CDN



(b) DDN-CDN



(c) DDN-DDN

Figure: Velocity profiles for $\nu = 0.041$, $\boldsymbol{f}(x_1, x_2) := (\sin(x_1) + \sin(x_2), 0)$ and different combinations of open boundary conditions.

Some references

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Braack, M., Mucha, P.B., Directional do-nothing condition for the Navier-Stokes equations, *J. Comput. Math.*, **32**(5) (2014), 507–521.

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Dong, S., A convective-like energy-stable open boundary condition for simulations of incompressible flows, *J. Comput. Phys.*, **302** (2015), 300-328.

Neustupa, T., Existence of steady flows of a viscous incompressible fluid through a profile cascade and their L^r-regularity., *Math. Meth. Appl. Sci.* 45 (2022), 4, 1827–1844.

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Problem: Analysis and numerical experiments of (NS+DDN).

Artificial boundary conditions for fluid flow simulation

Flow in exterior domains

Navier-Stokes flow past a translating body

$$\begin{cases} -\nu\Delta \boldsymbol{v} + ((\boldsymbol{v}-\boldsymbol{\xi})\cdot\nabla)\boldsymbol{v} + \nabla p = \boldsymbol{f} \text{ in } \Omega \subset \mathbb{R}^n,\ n \in \{2,3\} \\ \nabla \cdot \boldsymbol{v} = 0 \text{ in } \Omega \\ \boldsymbol{v} = \boldsymbol{g} \text{ on } \partial\Omega \\ \lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{0} \end{cases}$$

Approximation by flows in bounded domains

$$\begin{cases} -\nu\Delta \boldsymbol{v} + ((\boldsymbol{v}-\boldsymbol{\xi})\cdot\nabla)\boldsymbol{v} + \nabla p = \boldsymbol{f} \text{ in } \Omega_R \subset \mathbb{R}^3, \\ \nabla\cdot \boldsymbol{v} = 0 \text{ in } \Omega_R \\ \boldsymbol{v} = \boldsymbol{g} \text{ on } \partial\Omega \\ \nu \frac{\boldsymbol{x}}{R}\cdot\nabla\boldsymbol{v} - p\frac{\boldsymbol{x}}{R} + \frac{1}{R}\left(1+\mathfrak{s}(\boldsymbol{x})\right)\boldsymbol{v} - \frac{1}{2}\left(\boldsymbol{v}\cdot\frac{\boldsymbol{x}}{R}\right)\boldsymbol{v} = \boldsymbol{0} \text{ on } \partial B_R \end{cases}$$

$$\mathfrak{s}(\boldsymbol{x}) := \left[|\zeta||x| + (\zeta\cdot x)\right]/2$$

P. Deuring and S. Kračmar. Exterior stationary Navier-Stokes flows in 3D with non-zero velocity at infinity: approximation by flows in bounded domains. *Math. Nachr.*, 269/270:86–115, 2004.

Problem: Convergence when $R \to \infty$ and numerical experiments, starting with the Oseen case.

MMAC Specialization Profiles

Applied and Industrial Mathematics

Technomathematics

EUROPEAN CONSORTIUM FOR MATHEMATICS IN INDUSTRY

- Some Master Theses (concluded):

– Ana Cláudia Galhoz. "Simulation of liquid emptying from horizontal and inclined tubes with Smoothed Particle Hydrodynamics".
Co-supervisor: Arris Tijsseling, TU Eindhoven.

This thesis was carried out within the framework of the ECMI. Ana Galhoz was the first portuguese student to receive the ECMI certificate.

– Ricardo Melo Faria. "A fluid-structure interaction model for the study of earthquake response in a dam-water system".
Co-supervisor: Sérgio Oliveira, LNEC.