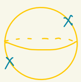
FROM GROMOV–WITTEN THEORY TO DYNAMICS.

Joint w. Julian Chaidez

Geometry

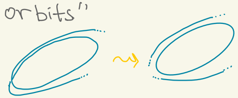
Gromov-Witten invariants ≈ "counts" of hol. curves w. constraints.



$\Rightarrow$

Dynamics

Question (Poincaré) on "closing almost closed orbits"

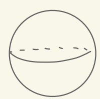


Hamiltonian dynamics:

- Symplectic manifold: (M^{2n}, ω)

  $\omega \in \Omega^2(M)$ closed, non-deg.

  (position–momentum space)



S^2

- $H : [0,1] \times M \to \mathbb{R}$, $\quad \omega(-, X_H) = dH$

  "energy" $\qquad$ symplectic gradient

- φ_H^t flow of X_H, $\varphi = \varphi_H^1$ "Hamiltonian diffeomorphism".

Example: $\mathbb{T}^n$ action on $\mathbb{CP}^n =: M$: $\forall a = (a_1, \dots, a_n) \in \mathbb{T}^n$

$$\varphi_a(z) := (a_1, \dots, a_n) \cdot [z_0 : \cdots : z_n] = [z_0 : e^{2\pi i a_1} z_1 : \cdots : e^{2\pi i a_n} z_n]$$

$\Rightarrow \varphi_a$ is Ham. diffeo.

$a \in \mathbb{Q}^n \Rightarrow \varphi_a$ per.

Non-example: translations on $\mathbb{T}^{2n}$.

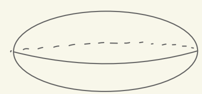
Reeb dynamics:

- Contact manifold: (Y^{2n-1}, α) with $\alpha \wedge (d\alpha)^n \neq 0$

- Reeb v.f. def by $\begin{cases} R \in \ker d\alpha \\ \alpha(R) = 1 \end{cases} \rightsquigarrow \varphi_\alpha^t$ "Reeb flow"

$$\boxed{\begin{array}{l} (\mathbb{R}_r \times Y, \ d(e^r \alpha)) \text{ symplectic} \\ \varphi_\alpha^t \text{ Ham. flow of } H(r,y) = r \end{array}}$$

Example: "ellipsoids" $\quad Y^{2n-1} = \left\{ z \in \mathbb{C}^n : \sum_{j=1}^n \frac{|z_j|^2}{a_j} = 1 \right\}$

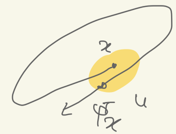


Reeb flow: $\varphi^t(z) = \left(e^{2\pi i t / a_1} z_1, \dots, e^{2\pi i t / a_n} z_n\right)$

The dynamical problem:

<u>Poincaré reccurence thm:</u> Let

- M closed + vol
- φ^t vol. pres flow (or diffeo)



then $\forall\, U \subseteq M$ and generic x,
$\exists\, T$ large s.t $\varphi^T x \in U$

<u>Question</u> (Poincaré) : Can we perturb φ^t to have a periodic orbit through U?

<u>Brief History</u>:

- Easy : C^0 pert of v.f. gen φ^t.
- 60-80's : Peixoto, Pugh, Pugh-Robinson
  C^1 — " — "Closing lemmas"
- 1991 : Herman: counter examples C^k $k \gg 1$, Ham, in dim ≥ 4.
- 2015 : Irie : C^∞ pert. for 3D Reeb flows.
  2016 : Asoaka-Irie : C^∞ Ham 2D.
  " C^∞ closing lemmas"

  } strong closing property

<u>Def</u> [Irie] • A Ham diffeo $\varphi : M \to M$ has the <u>strong closing property</u>

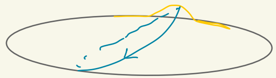
$\forall$ const $\not\equiv h : M \to \mathbb{R}_{\geq 0}$ $\exists\, \tau \in [0,1]$ s.t. $\underbrace{\varphi^1_{\tau h} \circ \varphi}_{\text{pertubed}}$ has a periodic point in $\mathrm{Supp}(h)$.

- A Reeb flow φ^t has the strong closing property if $\forall$ const $\not\equiv h : Y \to \mathbb{R}_{\geq 0}$ $\exists\, \tau \in [0,1]$ s.t. $\varphi^t_{e^{\tau h}\alpha}$ has per. orbit through $\mathrm{supp}(h)$.

Made with Goodnotes

Remark:

- strong closing $\Rightarrow C^\infty$
- local perturbations.



Y

Conjecture: [Irie, 2022]: The strong closing property holds for Reeb flows on ellipsoids. ANY DIM.

Proofs: • C-D-P-T "contact hom"

- Xue dyn methods.
- Çineli - Seyfaddini "Ham FH"

inspired by Fish-Hofer:

Thm [Chaidez-T.]

- Let (M^{2n}, ω) closed symp mfld s.t. $[\omega] \in H^2(M;\mathbb{Q})$ → ① Fails Herman ② true 2D.

  Then the strong closing property holds for "Hofer nearly periodic" (H.n.p) Ham. diffeos.

- Let Y closed contact mfld + Foundations for orbifolds → GW theory compactness.

  then the strong closing property "Hofer nearly periodic" Reeb flows.

Def. • Ham diffeo $\varphi = \varphi_H^1$ is H.n.p if $\exists\ \varphi_{h_j}^1$ periodic, i.e. $\varphi_{h_j}^{N_j} = \mathbb{1}$,

and $N_j \cdot \|h_j - H\|_{C^0} \longrightarrow 0$

• Reeb flow φ_α^t is H.n.p if $\exists$ $\varphi_{\alpha_j}^t$ periodic, ie $\exists\ T_j$ s.t. $\varphi_{\alpha_j}^{T_j} = \mathbb{1}$,

and $T_j \cdot \|\alpha_j - \alpha\|_{C^0} \longrightarrow 0$.

Example: • Ham $\mathbb{T}^k$ action M^{2n}

• Contact $\mathbb{T}^k$ actions on Y^{2n-1}.

Proof:

① Construct geometric invariants "spectral gaps".

detect the strong closing property

② GW invariants $\Rightarrow$ estimate Spectral gaps (for H.n.p.)

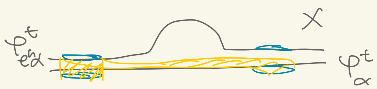
From dyn to geometry

(Reeb flows).

Y^{2n-1} contact, φ^t_α Reeb flow.

deformations cobordism $\subseteq \mathbb{R}\times Y$

$\varphi^t_\alpha \rightsquigarrow \varphi^t_{e^h\alpha}$ $\rightsquigarrow$



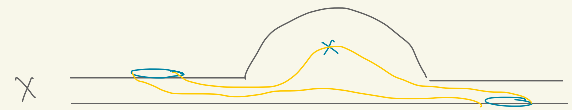
"subgraph of e^h".

X^{2n} symplectic. has almost complex str. $J: TX \to TX$, $J^2 = -\mathbb{1}$

<u>Hofer</u>: construct a.c.s J that "sees" the dynamic on ∂X.

$\exists$ hol. curve $\Longrightarrow$ $\exists$ per. orbit.

Point constraint



" large area"

Want: small area hol. curves through a point.

Spectral gaps $\approx$ $g(X) = \sup\limits_{J,P} \min\limits_{u} \text{area}(u)$

$\uparrow$ J-hol pass through P.

+ technical changes.

GW theory:

Enumerative geometry:

degree, #pts $\rightsquigarrow$ count inv of def.

GW inv:

Riem surface	(X,ω,J) closed,	homology $A \in H_2(X)$ (degree),	constraints (e.g. pts)
(Σ_g, j),			

$\rightsquigarrow$ "count" of $u : (\Sigma_g, j) \to (X, J)$

$$\underbrace{du \circ j = J \circ du, \quad u_*[\Sigma] = A, \quad u \text{ sat constraints.}}$$

$GW^{X,A}_{g,k} \in \mathbb{Q}$.

k: # constraints.

$GW \neq 0 \implies \exists$ such hol. curves.

Y, φ^t free S^1 action

$X = [0,1] \times Y$.

$[-\delta, 1+\delta] \times Y \big/ S^1 \curvearrowright \partial_\pm \supseteq X$