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RESOLUTION OF SINGULARITIES OF AN ALGEBRAIC VARIETY OVER A FIELD OF CHARACTERISTIC ZERO: II

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CHAPTER III. EFFECTS OF PERMISSIBLE MONOIDAL TRANSFORMATIONS ON SINGULARITIES.

1. The numerical characters $\nu^*(\mathbf{J})$ and $\nu(\mathbf{J})$ of a local ideal J, and a standard base of J

Let R be a regular local ring and J an ideal in R. Let M be the maximal ideal of R. We have defined the homogeneous ideal $\mathrm{gr}_{\mathbf{M}}(\mathbf{J}, \mathbf{R})$ in the graded R/M-algebra $\mathrm{gr}_{\mathbf{M}}(\mathbf{R})$.[1]

DEFINITION 1. *Given* R *and* J *as above, we define* $\nu^{(i)}(\mathbf{J})$, (*a non-negative integer or infinity,* ∞ *in symbol*) *for every positive integer* i *as follows:* $\nu^{(i)}(\mathbf{J})$ *is the maximal integer* ν, *if it exists, such that there exists a system of homogeneous elements* $(\varphi_1, \varphi_2, \cdots, \varphi_{i-1})$ *in* $\mathrm{gr}_{\mathbf{M}}(\mathbf{J}, \mathbf{R})$ *having the property that*

$$(\varphi_1, \cdots, \varphi_{i-1})\,\mathrm{gr}_{\mathbf{M}}(\mathbf{R}) \cap \mathrm{gr}^{\mu}_{\mathbf{M}}(\mathbf{R}) = \mathrm{gr}^{\mu}_{\mathbf{M}}(\mathbf{J}, \mathbf{R})$$

for all $\mu < \nu$; *and, if such* ν *does not exist, we set* $\nu^{(i)}(\mathbf{J}) = \infty$. (An empty system of elements generates the zero ideal.)

LEMMA 1. *Let* $(\varphi_1, \cdots, \varphi_m)$ *be a system of homogeneous elements of* $\mathrm{gr}_{\mathbf{M}}(\mathbf{J}, \mathbf{R})$ *such that*

(i) $\mathrm{gr}_{\mathbf{M}}(\mathbf{J}, \mathbf{R}) = (\varphi_1, \cdots, \varphi_m)\,\mathrm{gr}_{\mathbf{M}}(\mathbf{R})$,

(ii) *if* $\nu_i = \deg \varphi_i (1 \leqq i \leqq m)$, *then* $\nu_1 \leqq \nu_2 \leqq \cdots \leqq \nu_m$, *and*

(iii) *for every* $i \geqq 1$, $\varphi_i \notin (\varphi_1, \cdots, \varphi_{i-1})\,\mathrm{gr}_{\mathbf{M}}(\mathbf{R})$ (*where the empty system of elements generates the zero ideal*). *Then we have* $\nu^{(i)}(\mathbf{J}) = \nu_i$ *for* $1 \leqq i \leqq m$ *and* $\nu^{(i)}(\mathbf{J}) = \infty$ *for all* $i > m$.

PROOF. Let $\mu_i = \nu^{(i)}(\mathbf{J})$. In view of (i) and (ii), it is clear from Definition 1 that $\nu_i \leqq \mu_i$ for all i $(1 \leqq i \leqq m)$, and also that $\mu_i = \infty$ for all $i > m$. Suppose we have i $(1 \leqq i \leqq m)$ such that $\mu_i > \nu_i$. Let i be the smallest integer with this property. By Definition 1, we have homogeneous elements $\psi_1, \cdots, \psi_{i-1}$ such that

$$(\psi_1, \cdots, \psi_{i-1})\,\mathrm{gr}_{\mathbf{M}}(\mathbf{R}) \cap \mathrm{gr}^{\mu}_{\mathbf{M}}(\mathbf{R}) = \mathrm{gr}^{\mu}_{\mathbf{M}}(\mathbf{J}, \mathbf{R})$$

for all $\mu < \mu_i$. We may assume that $\deg \psi_1 \leqq \cdots \leqq \deg \psi_{i-1}$. Then, for every $j < i$, we have

$$(\psi_1, \cdots, \psi_{j-1})\,\mathrm{gr}_{\mathbf{M}}(\mathbf{R}) \cap \mathrm{gr}^{\mu}_{\mathbf{M}}(\mathbf{R}) = \mathrm{gr}^{\mu}_{\mathbf{M}}(\mathbf{J}, \mathbf{R})$$

[1] cf. §2, Chap. II.