Big fiber theorems and symplectic rigidity

Geometria em Lisboa

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Based on joint work together with Adi Dickstein, Yaniv Ganor, Leonid Polterovich

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- $f\colon \Delta^{p(d+1)} \to \mathbb{R}^d$ continuous $\Rightarrow$ f has a fiber intersecting all pd-dimensional faces (Karasev 2012). For affine f proved by Rado and Neumann circa 1945.

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- IVMs can be extended to compact subsets by $\mu(K) = \bigcap_{U \supset K} \mu(U)$; this satisfies analogous properties.

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$$\mu(\mathbb{C}P^k) = (h^{n-k}) \subset \mathbb{F}[h]/(h^{n+1})\,.$$

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This defines an A-IVM on X. Passage to compacts commutes with pushforwards for compact metrizable spaces.

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Consider Φ: $\mathbb{C}P^n \to \mathbb{R}^n$: $\Phi([z_0 : \cdots : z_n]) = \frac{1}{|z|^2}(|z_1|^2, \ldots, |z_n|^2)$; $\Phi(\mathbb{C}P^n) = \Delta^n$.
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$$(f_*\nu)(Z) = \nu(f^{-1}(Z)) = \nu(f^{-1}(f(D))) \supset \nu(D) = I\,,$$

by monotonicity and since $f^{-1}(f(D)) \supset D$.

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Let $f\colon \Delta^{p(d+1)} \to Y$ be a continuous map, where Y is a metrizable space of covering dimension d. Let $\nu = \Phi_*\mu$ be the $H^*(\mathbb{C}P^{p(d+1)})$-IVM on $\Delta^{p(d+1)}$. Let $I = (h^p) \in \mathcal{I}(H^*(\mathbb{C}P^{p(d+1)}))$. Then $I^{\smile(d+1)} = (h^{p(d+1)}) \neq 0$. The abstract centerpoint theorem applies to $f_*\nu$ and yields $y_0 \in Y$ such that $Z \subset Y$ compact, $I \subset (f_*\nu)(Z) \;\Rightarrow\; y_0 \in Z$. Let $D \subset \Delta^{p(d+1)}$ be a *pd*-dimensional face and let $Z = f(D)$. Then $\nu(D) = (h^p) = I$, and thus

$$(f_*\nu)(Z) = \nu(f^{-1}(Z)) = \nu(f^{-1}(f(D))) \supset \nu(D) = I\,,$$

by monotonicity and since $f^{-1}(f(D)) \supset D$. It follows that $y_0 \in Z = f(D)$, or equivalently $D \cap f^{-1}(y_0) \neq 0$. □

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Two closed (respectively, open) subsets A, A' of a closed symplectic manifold (M, ω) **commute** if there is an involutive map $\pi \colon M \to B$ and closed (respectively, open) sets $C, C' \subset B$ such that $A = \pi^{-1}(C)$, $A' = \pi^{-1}(C')$.

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- **(vanishing):** if $K \subset M$ is a displaceable compact, then $\tau(M \setminus K) = A$, and there is open $U \supset K$ with $\tau(U) = 0$.

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Theorem (DGPZ 2021)

Any closed symplectic manifold (M, ω) carries a $QH^*(M)$-IVQM which also satisfies invariance and vanishing.

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Example

On S^2, the quantum cohomology IVQM τ can be described as follows: for an open set $U \subset S^2$, $\tau(U) = 0$ if for every compact connected smooth subsurface $Q \subset U$ there exists a smooth closed disk of area $< \frac{1}{2}$ containing Q. Otherwise $\tau(U) = QH^*(S^2)$.

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The abstract centerpoint theorem applied to the A-IVM $\pi_*\tau$, yields $b_0 \in B$ such that for each compact $C \subset B$ with $I \subset (\pi_*\tau)(C)$ satisfies $b_0 \in C$. If $Z \subset M$ is compact and $I \subset \tau(Z)$, then $\pi^{-1}(\pi(Z)) \supset Z$ and thus $(\pi_*\tau)(\pi(Z)) = \tau(\pi^{-1}(\pi(Z))) \supset \tau(Z) \supset I \Rightarrow b_0 \in \pi(Z) \Rightarrow \pi^{-1}(b_0) \cap Z \neq \varnothing$. □

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