EXPLORING THE BOUNDARY OF THE MODULI SPACE OF STABLE SURFACES: SOME EXPLICIT EXAMPLES

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§ 0 - Notation

We work over $\mathbb{C}$.

if X proj. variety, ω_X = dualizing sheaf.

- ω_X is a rk 1 torsion free sheaf

- X smooth $\Rightarrow$ $\omega_X = \Lambda^{\dim X} \Omega^1_X$ is a line bundle, called canonical bundle

- X is Gorenstein if ω_X is a line bundle, e.g. if X is locally a hypersurface

§ 1 - (Compactified) moduli spaces

$g \geq 2$. M_g = moduli space of smooth proj. curves of genus g.

M_g is irred quasi proj. variety, $\dim = 3g-3$, mildly singular

Rem: C smooth proj curve

$g(C) \geq 2 \iff \omega_C$ is ample

$\exists$ modular compactification:

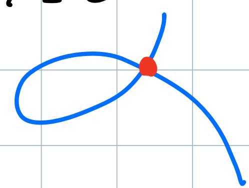
$$\mathcal{M}_g \hookrightarrow \overline{\mathcal{M}}_g$$

$\overline{\mathcal{M}}_g$ = moduli space of stable curves of genus g

C stable curve if:

- the sing. of C are nodes, i.e. locally $xy=0$

  $\rightsquigarrow$ C is Gorenstein

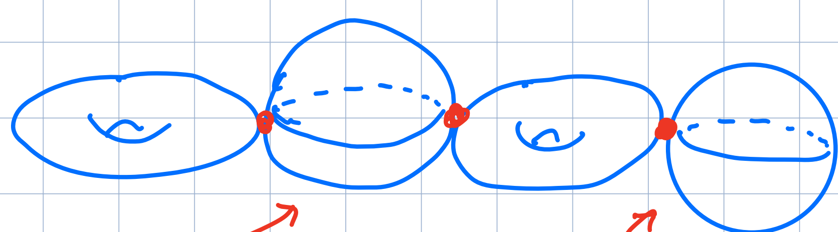


- ω_C is ample.

Example:

C nodal but not stable:

ω_C is trivial here



ω_C negative here

$\overline{M}_g$ is an irred. projective variety of dim $3g-3$

(Deligne – Mumford, 1969)

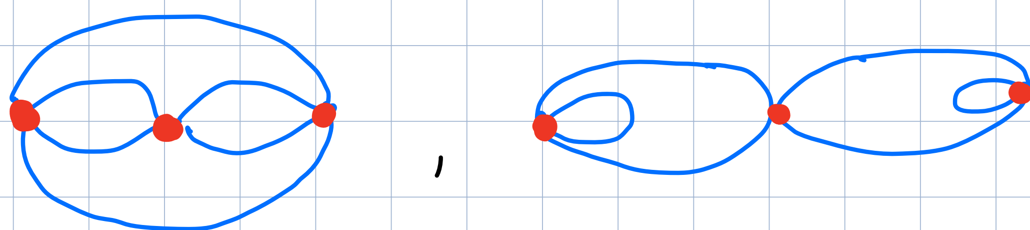
The boundary $\overline{M}_g \setminus M_g$ can be explicitly described.

<u>Example</u>: $g = 2$, $\dim \overline{M}_3 = 3$

cod 1 :

cod 2 :

cod 3 :



Surfaces: S smooth proj. surface of gen type (i.e. $\omega_S^{\otimes n}$ gives a birational map for $n \gg 0$)

$\exists!$ smooth (or mildly singular) surface X birational to S and such that ω_X is ample.

X = canonical model of S

Numerical invariants:

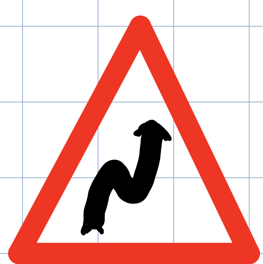
- $\chi(X) := \chi(\mathcal{O}_X) = 1 - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) \in \mathbb{N}_{>0}$
$$= 1 - h^0(\Omega^1_X) + h^0(\omega_X)$$
$q(X)$, irregularity
$p_g(X)$, geometric genus

- $K_X^2 := c_1(\omega_X)^2 \in \mathbb{N}_{>0}$

For fixed $a, b \in \mathbb{N}_{>0}$,

$\mathcal{M}_{a,b}$ = moduli space of (the canonical models of) surfaces of general type with $K^2 = a$, $\chi = b$ (Gieseker 1979)

$\mathcal{M}_{a,b}$ is quasi-projective, not projective



$\mathcal{M}_{a,b}$ is not as well behaved as $\mathcal{M}_g$!

Important differences:

- $M_{a,b}$ can be empty!
(we only have a lower bound for the dimension)
- $M_{a,b}$ can have arbitrarily many irred./connected comp. of different dimensions
- any possible sing. occurs on some $M_{a,b}$ (Vakil's "Murphy's law")

$\exists$ Modular compactification:

$$\mathcal{M}_{a,b} \hookrightarrow \overline{\mathcal{M}}_{a,b} \quad \text{projective}$$

$\overline{\mathcal{M}}_{a,b}$ = moduli space of stable surfaces with $K^2 = a$, $\chi = b$

Long history:

- 1991: Kollár - Shepherd Barron introduce stable surfaces
- 1994: Alexeev proves boundedness

⋮

- 2022: book by Kollár (with Altmann and Kovács) settles the theory for pairs in any dimension.

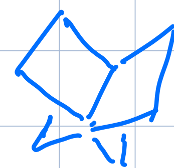
Stable surface: X proj. surf.

- $\exists$ $\Sigma \subseteq X$ finite set such that:

locally near $p \in X_0 = X \setminus \Sigma$

X is smooth or double crossings:

$$xy = 0$$



- sings of Σ are "not too bad",

i.e. slc ("semi-log canonical")

($\exists$ list of slc surface

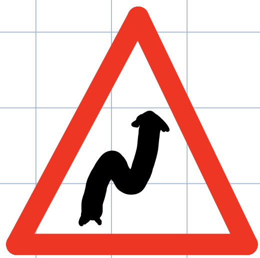
singularities)

- $\exists$ $m > 0$: $\omega_{X_0}^{\otimes m}$ extends to

a line bundle, denoted by

$\omega_X^{[m]}$, and $\omega_X^{[m]}$ is ample.

"X is $\mathbb{Q}$-Gorenstein"



(1) the definition of the functor corresponding to $\overline{\mathcal{M}}_{a,b}$ is very complicated.

Special case: if B is a curve, a family of stable surfaces is: $p : \mathcal{X} \longrightarrow B$ s.t.:

- p is flat
- X_b is a stable surface $\forall b \in B$
- $\mathcal{X}$ is $\mathbb{Q}$-Gorenstein

(2) $\mathcal{M}_{a,b}$ is open in $\overline{\mathcal{M}}_{a,b}$ but possibly not dense.

Questions:

- what stable surfaces do occur in $\overline{\mathcal{M}}_{a,b} \setminus \mathcal{M}_{a,b}$?

- which of the above admit a $\mathbb{Q}$-Gorenstein smoothing?
  i.e., correspond to points in the closure of $\mathcal{M}_{a,b}$.

Very few examples have been worked out.

§ 2 - I surfaces, a test case

I-surface: X minimal of gen type

$K_X^2 = 1$, $\chi = 3$. $(\Rightarrow p_g = 2, \ q = 0)$

Also called $(1,2)$-surfaces.

Why these?

- they appear as exc. to birationality of pluricanonical maps of surfaces and 3-folds (analogues of genus 2 curves
- first "interesting case" from the Hodge theoretic point of view
- very good grasp of the moduli space

- smallest possible K^2 and largest p_g for $K^2=1$

Long term project: understand $\overline{\mathfrak{M}}_{1,3}$

<u>Our team:</u>

$C :=$ S. Coughlan

$F :=$ M. Franciosi

$P :=$ –

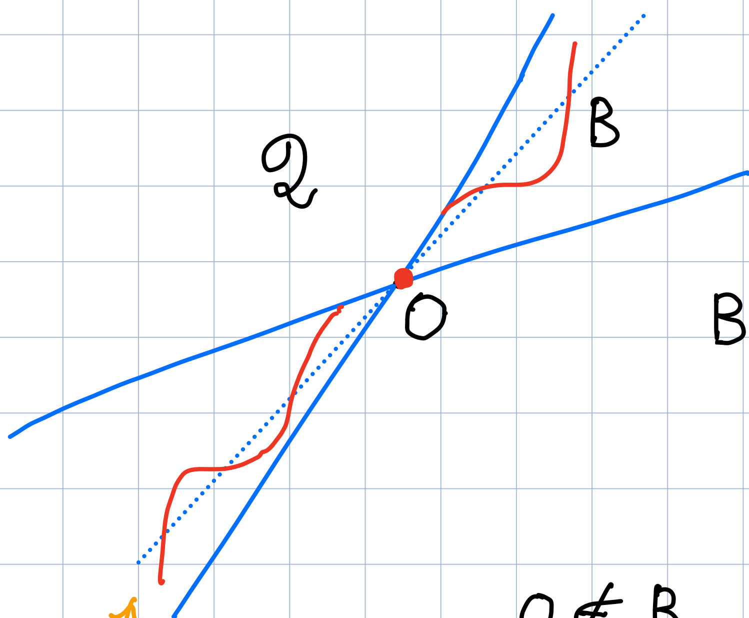
$Ra :=$ J. Rana

$Ro :=$ S. Rollenske

X an I-surface:

$$\varphi = \varphi_{2K_X} : X \xrightarrow{2:1} Q \subseteq \mathbb{P}^3$$

$z^2 = x^2 + y^2$, quadratic cone



$B = Q \cap \{F_5 = 0\}$

degree 5 hypersurf.

$O \notin B$

rulings pull back to canonical curves, $g = 2$

$\Rightarrow$ moduli space irred. of dim 28

very large!

Can describe various subsets of $\overline{\mathfrak{M}}_{1,3}$:

- Gorenstein I-surfaces (Cartier index 1) (FPRo 2017)
  open in $\mathfrak{M}_{1,3}$
- I-surfaces with one T-sing, (FPRaRo 2021, CFPRaRo 2022)
- I-surfaces with Cartier index 2: (CFPRo, in progress)
- Gallardo, Pearlstein, Schaffler, Zhang (2022): describe 8 new divisors in the closure of $\mathfrak{M}_{1,3}$ in $\overline{\mathfrak{M}}_{1,3}$
- Rollenske-Torres (2023): refinement of the above

Cyclic quotient sing: $a, m \in \mathbb{N}_{>0}$

$\xi \in \mathbb{C}$ a primitive m-th root of 1

$\mathbb{Z}_m$ acts on $\mathbb{C}^2$ by

$$(x,y) \longrightarrow (\xi x, \xi^a y)$$

$\mathbb{C}^2/\mathbb{Z}_m$ is a $\frac{1}{m}(1,a)$ sing

Def: a sing of type

$\frac{1}{n^2 d}(1, nda-1)$ with $(n,a)=1$

is a T-singularity

Example: $\frac{1}{4}(1,1)$ is the cone over the rational normal curve of degree 4. Resolved by a -4 curve

Kollar-Shepherd-Barron:

- RDP's and T-sings are the only quotient sing. that admit a $\mathbb{Q}$-Gorenstein smoothing.
- d = dimension of $\mathbb{Q}$-Gorenstein local def's

Namely: (1) expect these sing's on smoothable stable surfaces

(2) expect them in cod d

(for $d=1$, we expect a divisor)

§ 3 – I surfaces with 1 T singularity

FPRaRo 21 + CFPRaRo 22:

Possibilities for I surface with 1 T-sing: $\frac{1}{4}(1,1)$, $\frac{1}{18}(1,5)$, $\frac{1}{25}(1,14)$

$d=1$

$d=2$

$\frac{1}{4d}(1, 2d \overset{+}{-} 1)$, $d \leq 31$

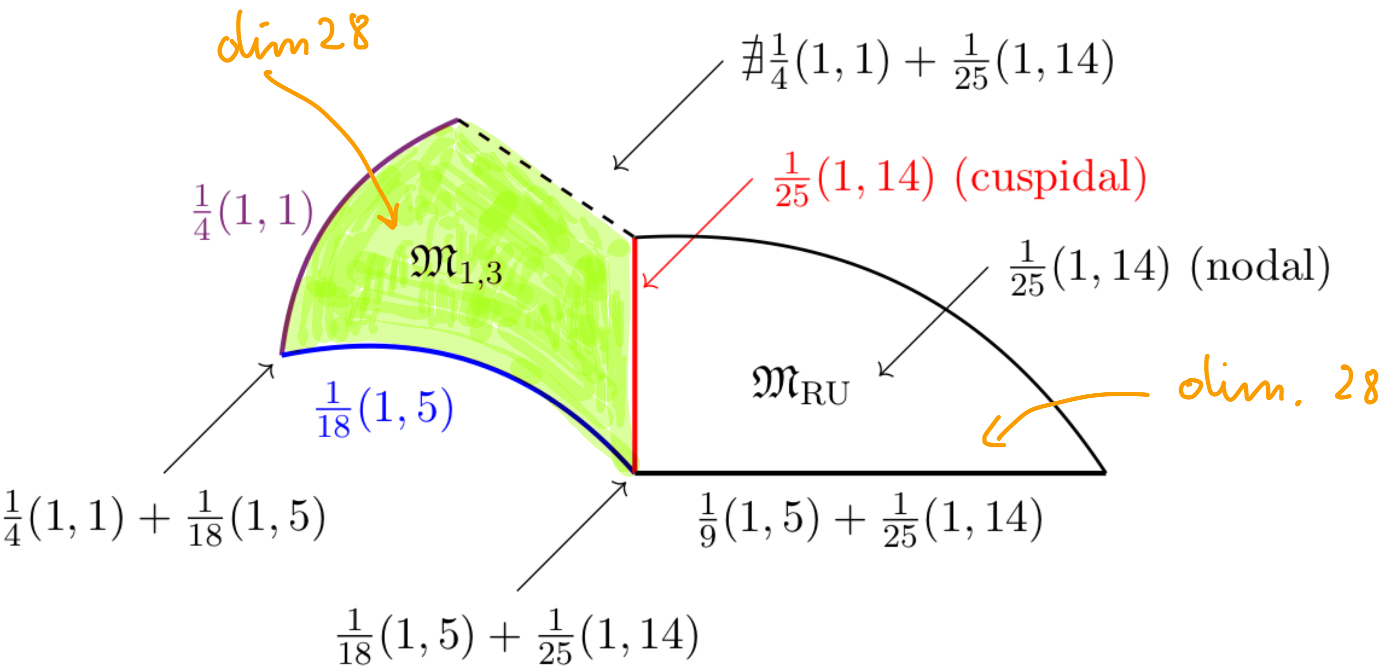


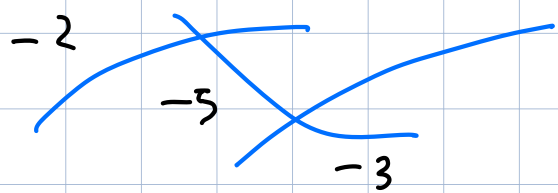
Figure 1. Schematic picture of (known parts of) the moduli space of I-surfaces

RU (Rana-Urzua) surfaces

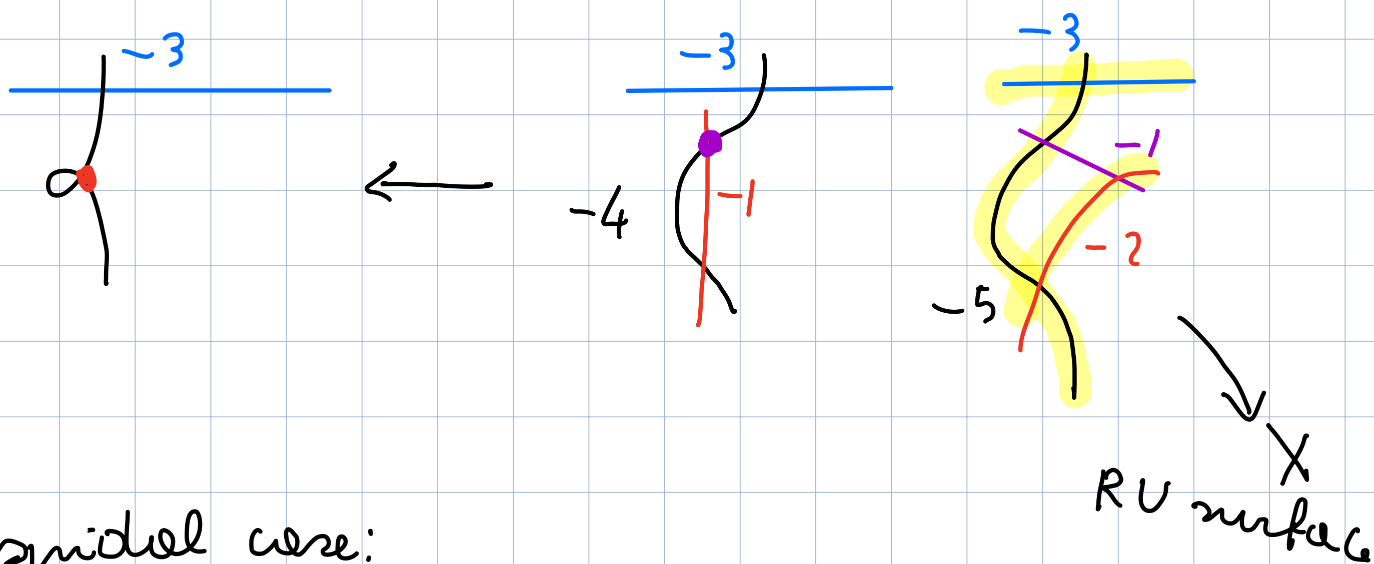
$\frac{1}{25}(1,14)$; $\frac{25}{14} = 2 - \frac{3}{14} = 2 - \cfrac{1}{5 - \cfrac{1}{3}}$

$\tilde{X} \longrightarrow X$ min. res,

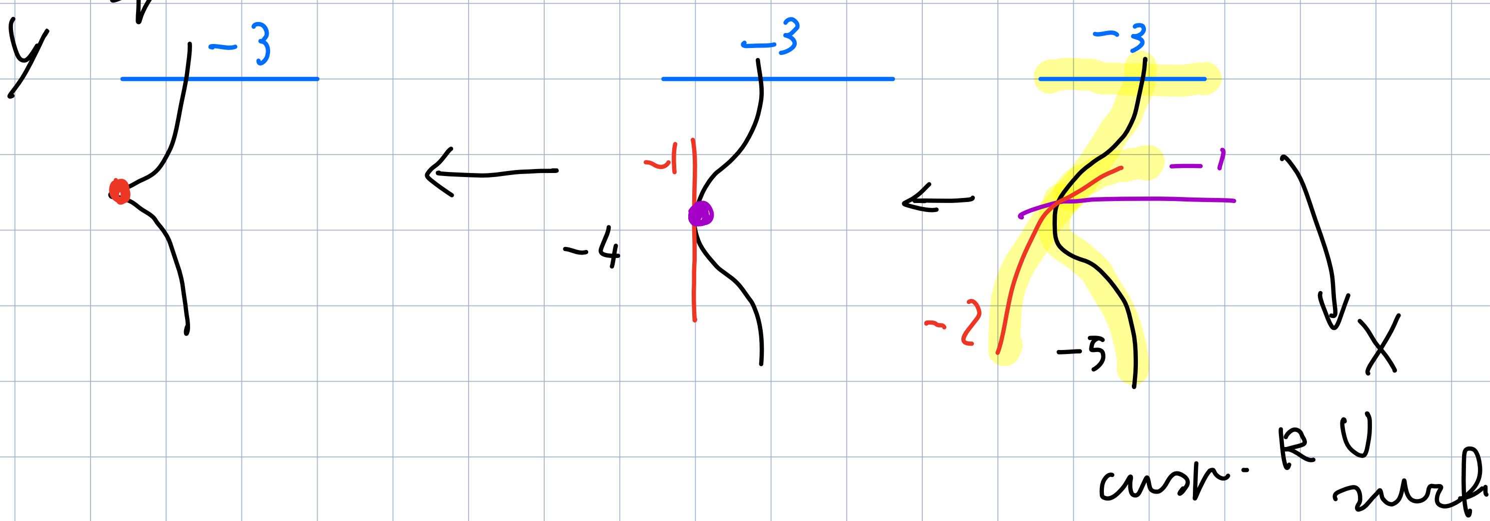
exc. divisor:



Y elliptic.



cuspidal case:



Gorenstein I-surfaces:

Thm (FPRo 17): all Gorenstein I-surfaces are double covers $X \to Q$ branched on $B \not\ni 0$, and $(Q, \frac{1}{2}B)$ lc

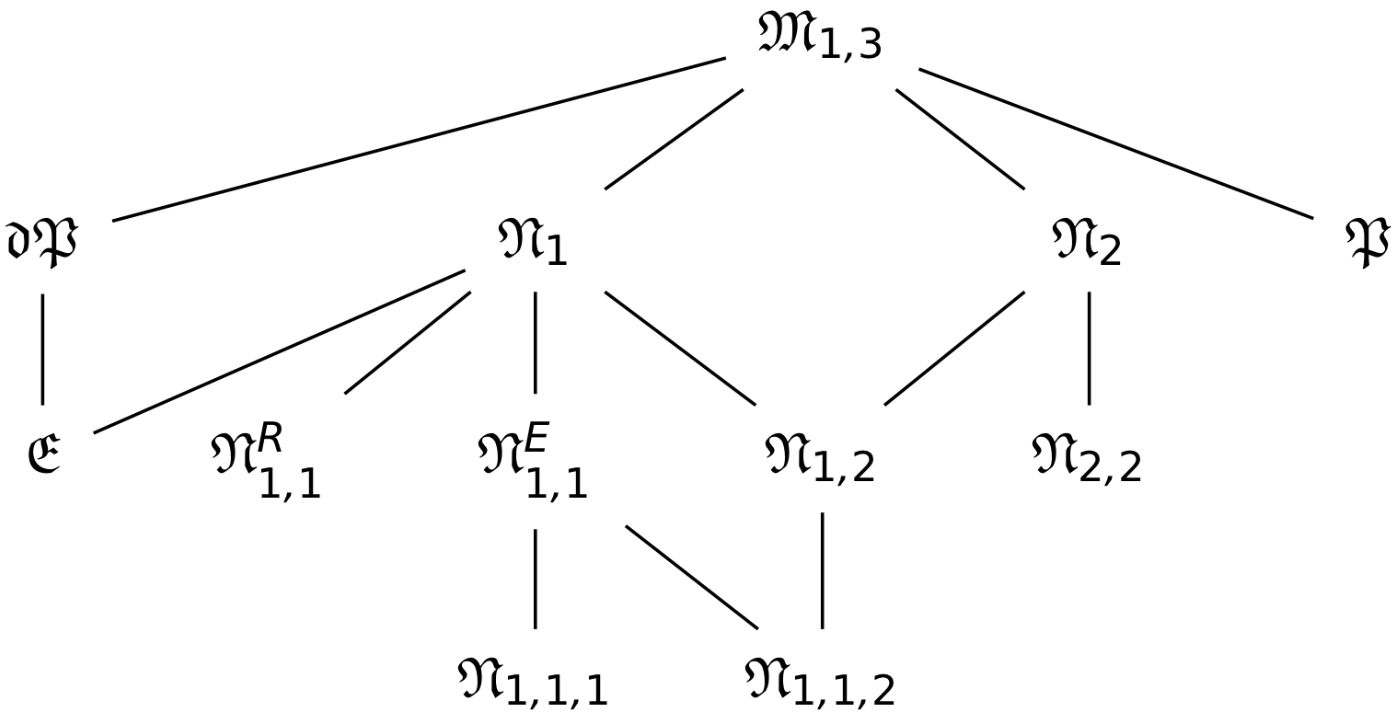
Stratification of $\overline{\mathfrak{M}}_{1,3}^{Gor}$

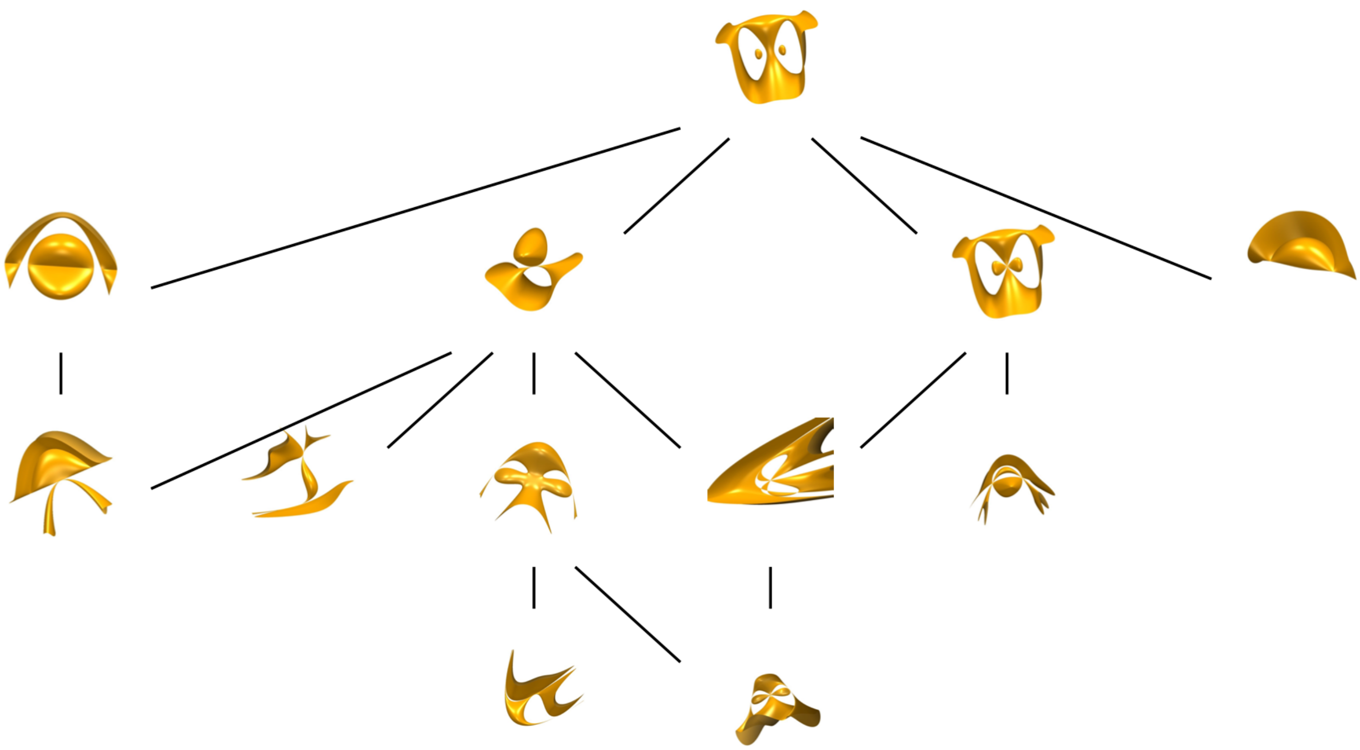
Stratum	Dimension	minimal resolution $\tilde{X}$	$\kappa(\tilde{X})$
$\mathfrak{N}_\varnothing = \mathfrak{M}_{1,3}$	28	general type	2
$\mathfrak{N}_2$	20	blow up of a K3-surface	0
$\mathfrak{N}_1$	19	minimal elliptic surface with $\chi(\tilde{X}) = 2$	1
$\mathfrak{N}_{2,2}$	12	rational surface	$-\infty$
$\mathfrak{N}_{1,2}$	11	rational surface	$-\infty$
$\mathfrak{N}_{1,1}^R$	10	rational surface	$-\infty$
$\mathfrak{N}_{1,1}^E$	10	blow up of an Enriques surface	0
$\mathfrak{N}_{1,1,2}$	2	ruled surface with $\chi(\tilde{X}) = 0$	$-\infty$
$\mathfrak{N}_{1,1,1}$	1	ruled surface with $\chi(\tilde{X}) = 0$	$-\infty$
$\mathfrak{dP}$	11	del Pezzo surface of degree 1	$-\infty$
$\mathfrak{P}$	4	$\mathbb{P}^2$	$-\infty$
$\mathfrak{E}$	2	minimal ruled surface with $\chi(\tilde{X}) = 0$	$-\infty$

- all smoothable

Rem:
- $\exists$ Hodge theoretic interpretation (Green-Griffiths-Laza-Robles)
- codimension of strata $\gg 0$

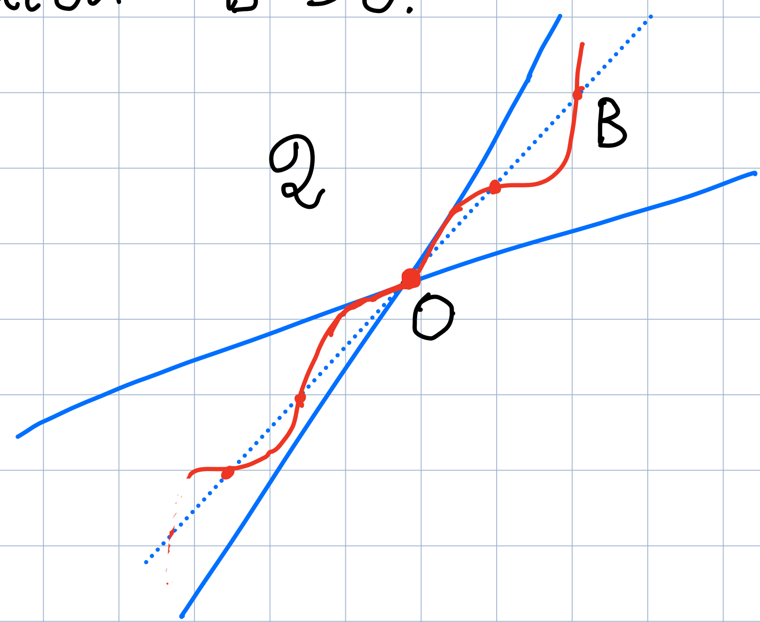
Tree of degenerations





I-surfaces of Cartier index 2

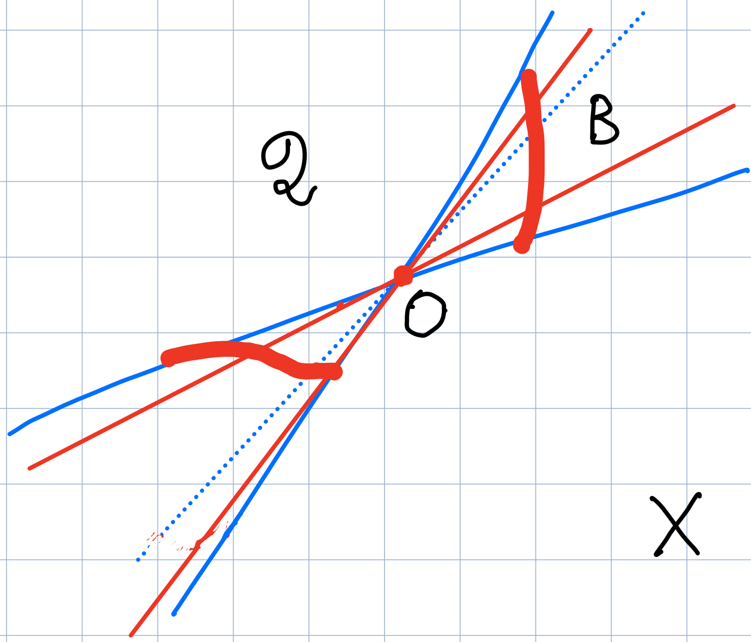
- double cover of cone branched on a quintic section $B \ni 0$:



if B general, $\frac{1}{4}(1,1)$ sing. over 0

$\Longrightarrow$ get a codimension 1 stratum of the boundary

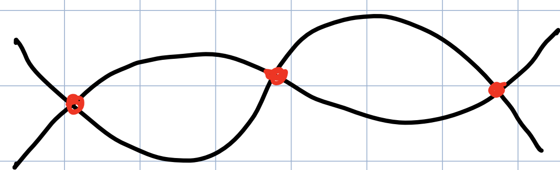
More degenerate examples occur this way:

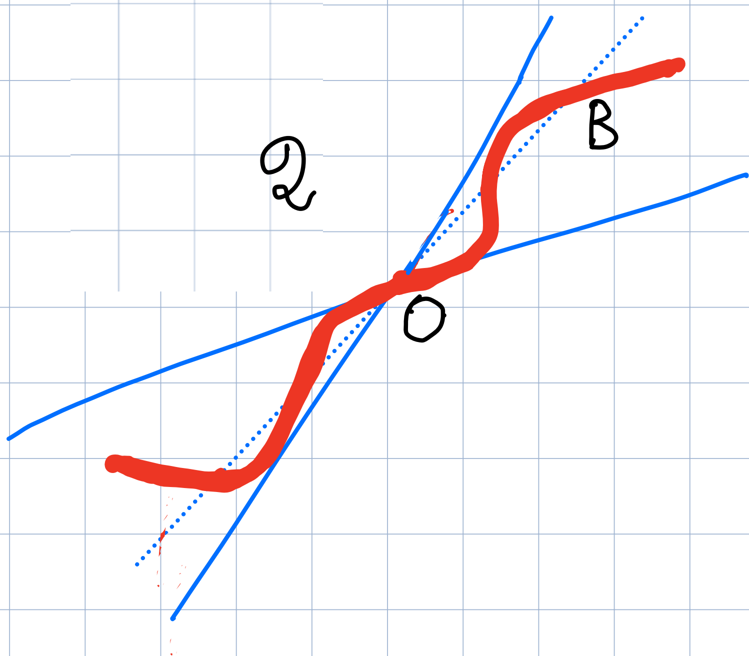


B = 2 rulings + 2× bisection

X irreducible

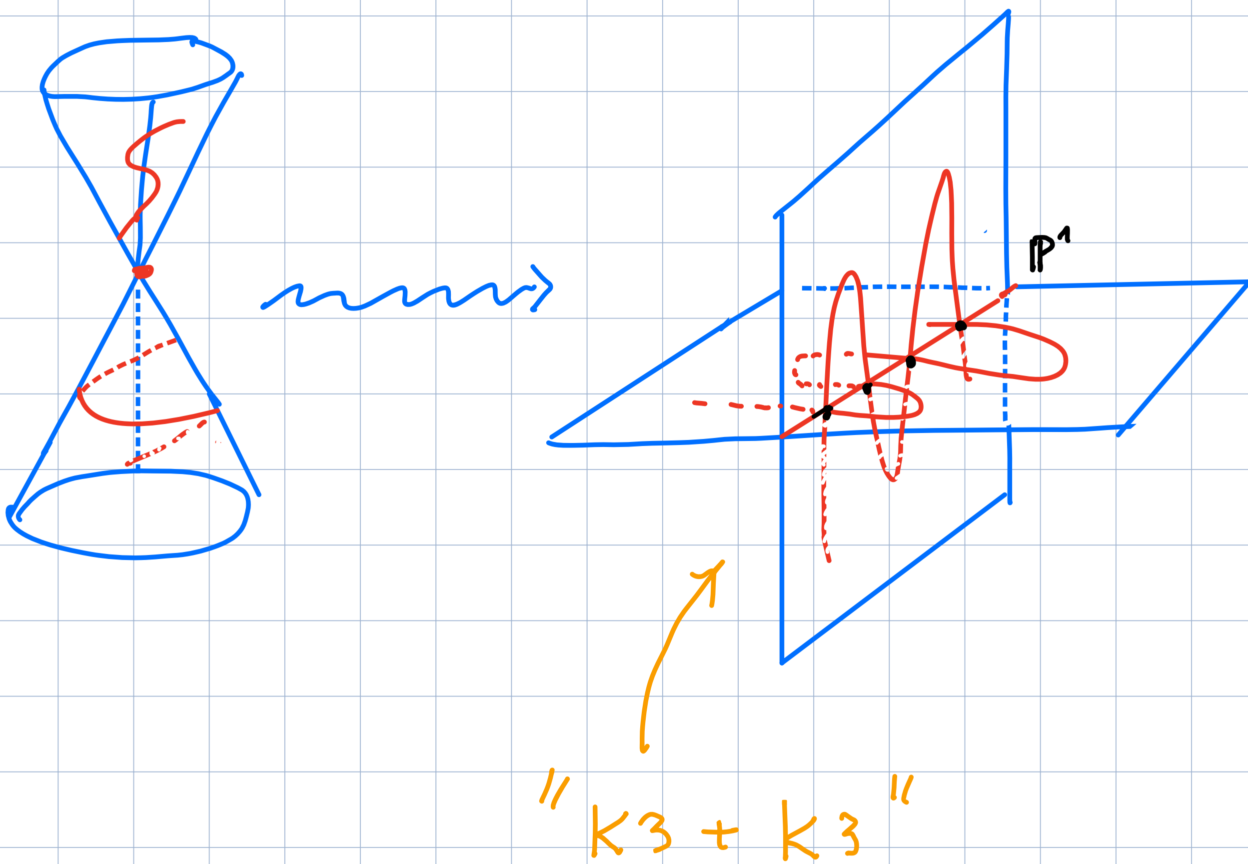
Canonical curve:





X reducible

- Degenerate Q to a pair of planes



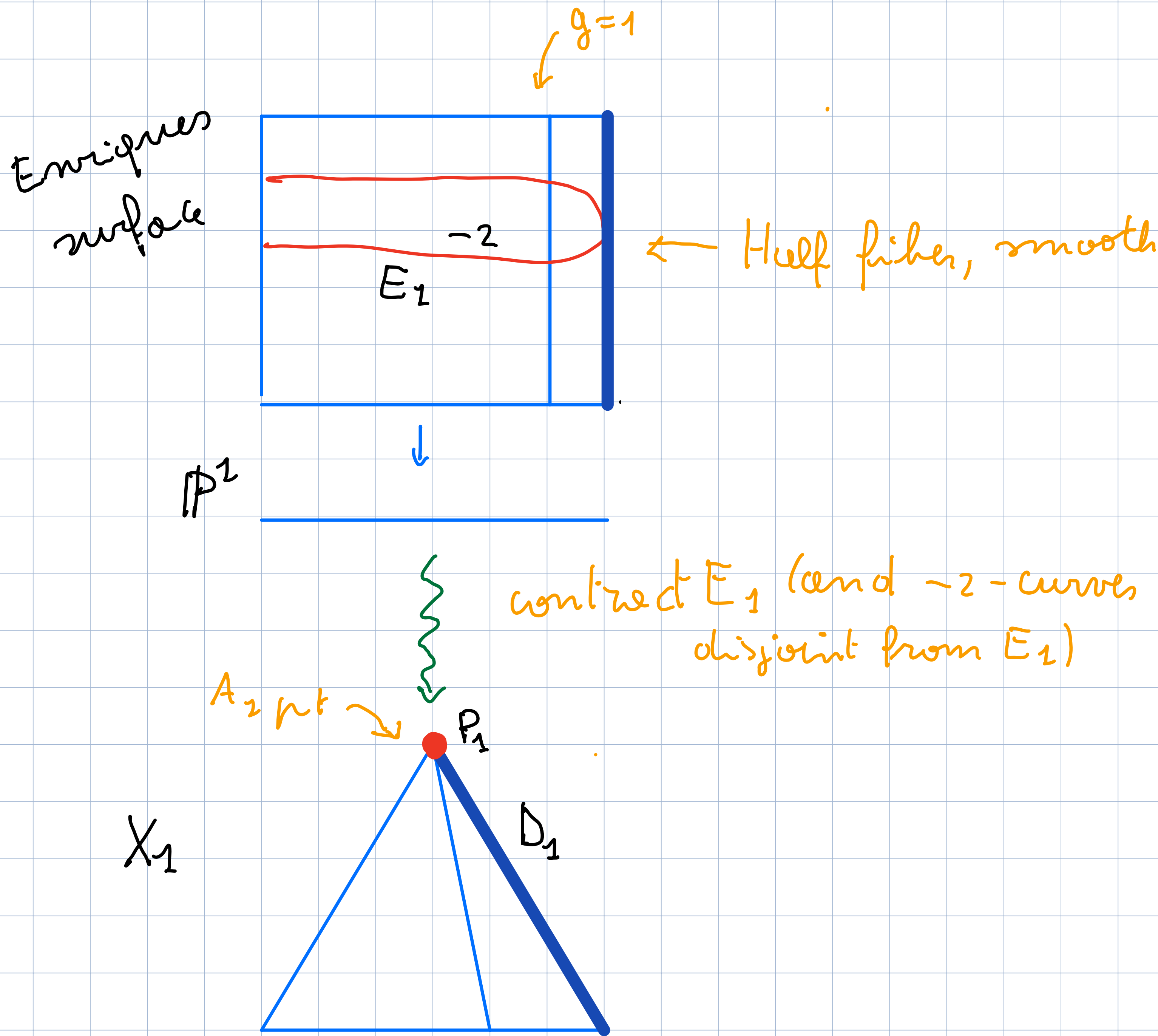
Union of 2 K3 surfaces with 5 A_1 points, glued along a $\mathbb{P}^1$

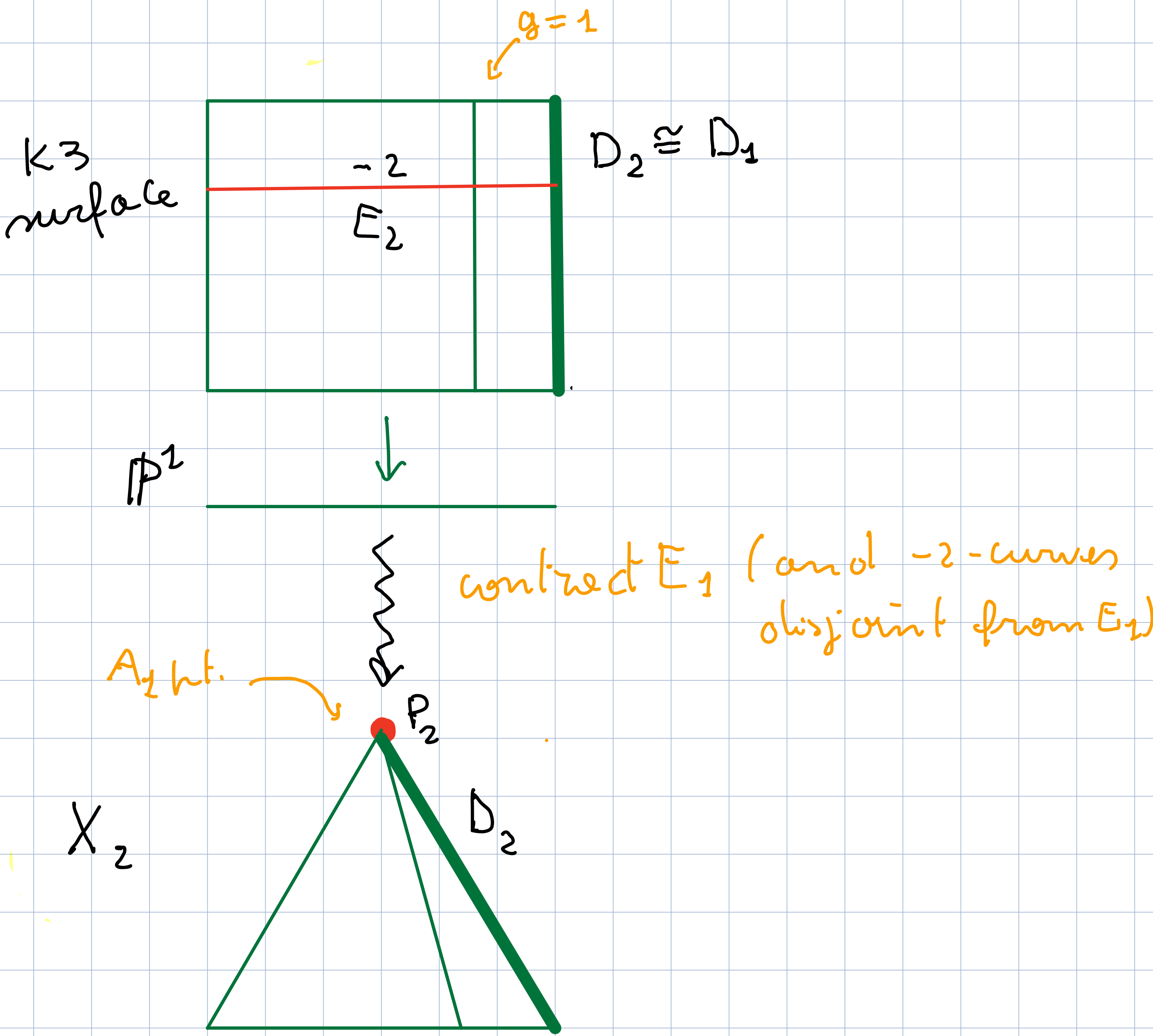
All the canonical curves are non reduced, with support $= \mathbb{P}^1$

24 dimensional family

All the above examples are clearly smoothable.

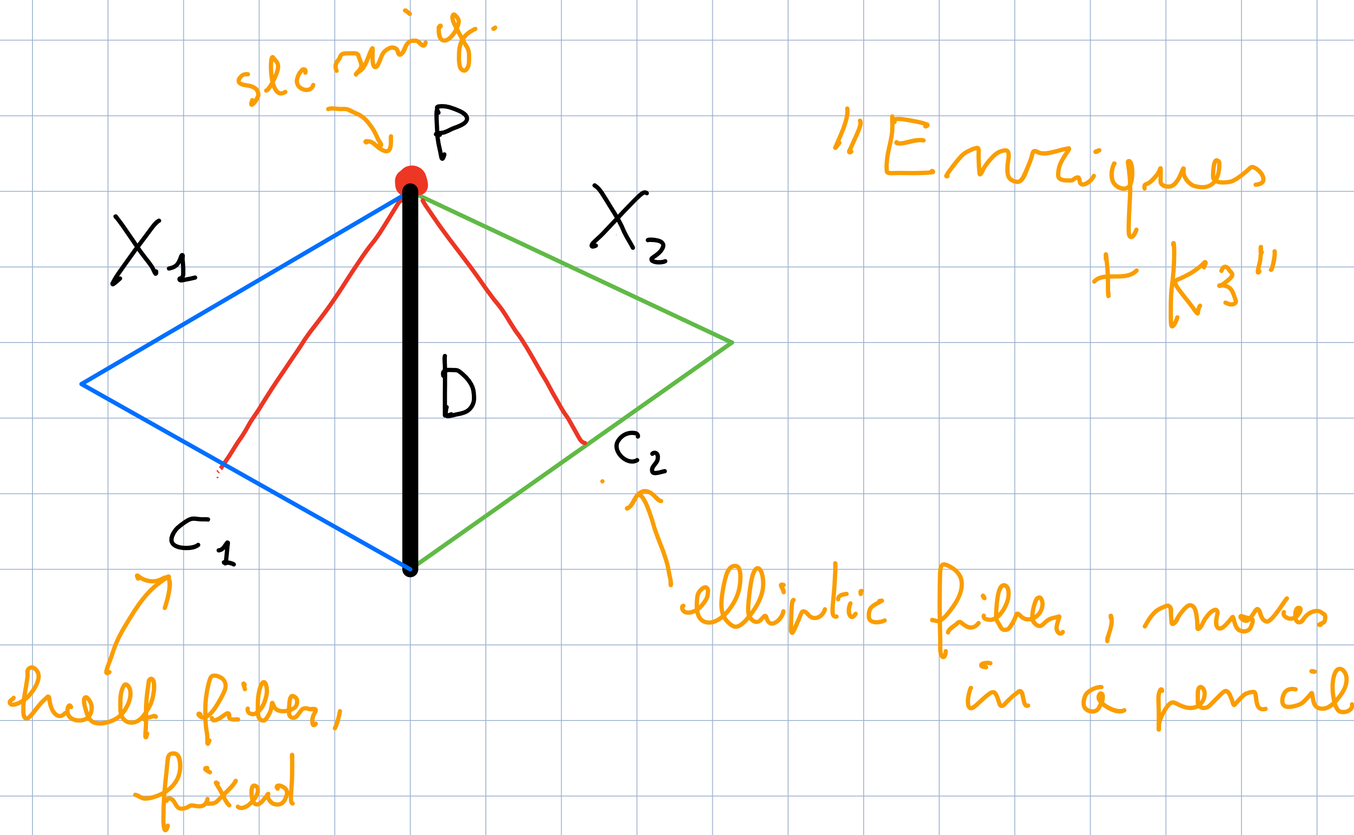
A different construction:





$$\varphi : D_1 \xrightarrow{\sim} D_2 \quad , \quad \varphi(P_1) = P_2$$

$X := X_1 \cup_\varphi X_2$ is a stable I-surface of index 2



$C = C_1 \cup C_2$ is a canonical curve

Moduli: 9 for the Enriques + 18 for the K3 = 27 ← cod 1

Theorem (CFPRo 23)

X an I-surface of index 2

$C \in |K_X|$ general. If C is reduced,

X is one of the following:

(A) double cover of cone Q

(B) "Enriques + K3"

In either case X is smoothable.

Conjecture (work in progress):

X I-surface of index 2, with

non reduced canonical curves

$\Rightarrow$ (a) X is "K3 + K3" (smoothable)

(b) X belongs to a 30-dim family of red. surfaces, both comp. are also K3.

What else is known about explicit compactifications?

Not much:

- Burniat surfaces with $K^2 = 6$ and Campedelli surfaces with $\pi_1 = \mathbb{Z}_2^3$. (Alexeev - P)
- some partial results for
  - $K^2 = 1$, $p_g = 0, 1$.
    (FPR, Franciosi Pardini Rollenske - Alm Ti Do)
  - $K^2 = 2$, $p_g = 3$, Gorenstein (Anthes)
- Horikawa surfaces.
  Rana Rollenske