Skein Theory and the Geometric Langlands Program

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Includes joint work with:

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• Skein modules

Definition: (Przytycki, Turaev)

Let M be a [oriented] 3-manifold

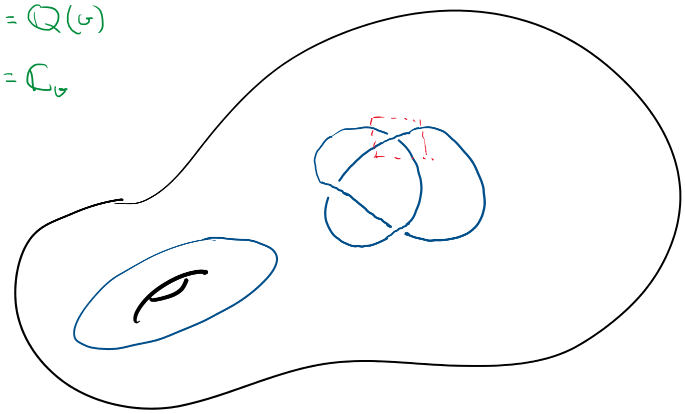
R a ring with a fixed element $v \in R^{\times}$.

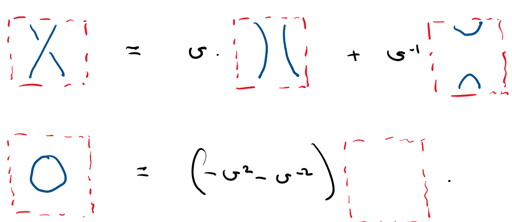
e.g. $R = \mathbb{Z}[v, v^{-1}]$

$R = \mathbb{Q}(v)$

$R = \mathbb{C}_v$

$Sk(M)$ is defined to be the R-module formally spanned by isotopy classes of framed links in M modulo the relations





Kauffman bracket skein relations.

Examples:

① (Jones, Kauffman) $Sk(\mathbb{R}^3) \cong R$

$$[L] \longmapsto K_L(v) \in \mathbb{Z}[v, v^{-1}].$$

[Can think of $Sk(M)$ as an abstraction to defining a polynomial invariant of knots in M.]

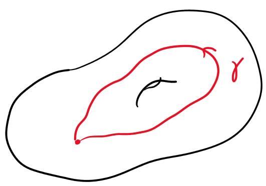
(2) (Hoste - Przytycki) $Sk(S^2 \times S^1) \cong R \oplus \bigoplus_{i=1}^{\infty} R/(1-v^{2i+4})$

Theorem (Bullock, Przytycki-Sikora)

$$Sk_{\mathbb{C}_{v=\pm1}}(M) \cong \mathbb{C}[Loc_{SL_2}(M)] = \mathbb{C}[Hom(\pi_1(M), SL_2)]^{SL_2}$$

$\mathbb{R}$

$$\langle \gamma \rangle \longmapsto [\tau_\gamma : \rho \longmapsto tr(\rho(\gamma))]$$



stack

Examples: (1) If $M = S^1 \times S^2$, $Loc_G(M) = SL_2 /_{conj} SL_2$

$$\mathbb{C}[Loc_{SL_2}(M)] = \mathbb{C}[SL_2]^{SL_2} = \mathbb{C}[x]$$

(2) If $M = T^3 = S^1 \times S^1 \times S^1$,

$$Loc_{SL_2}(M) = \left\{ (A_1, A_2, A_3) \in SL_2^3 \;\middle|\; \begin{matrix} A_1A_2 = A_2A_1 \\ A_1A_3 = A_3A_1 \\ A_2A_3 = A_3A_2 \end{matrix} \right\} / SL_2(\mathbb{C})$$

Theorem (Carrega, Gilmer ~'16)

$$\dim_{\mathbb{Q}(v)} Sk_{\mathbb{Q}(v)}(T^3) = 9$$

"generic v"

Theorem (G - Jordan - Safronov ~'19/'22)

If M is $\boxed{\text{closed}}$, then

$$\dim_{\mathbb{Q}(v)} Sk_{\boxed{\mathbb{Q}(v)}}(M) < \infty$$

generic v

Theorem (Gilmer - Masbaum, Detcherry - Wolfe)

$$\dim_{\mathbb{Q}(v)} Sk_{\mathbb{Q}(v)}(\Sigma_g \times S^1) = 2^{2g+1} + 2g - 1.$$

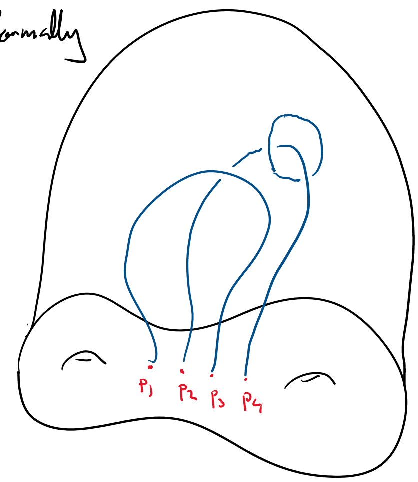
• The skein TQFT

Relative skein modules.

Definition: Let M be a 3-manifold (and $\{p_1, \dots, p_k\} \subseteq \partial M$)

R a ring with a fixed element $v \in R^\times$.

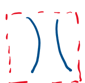
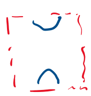
$Sk(M; \{p_1, \dots, p_k\})$ is defined to be the R-module formally

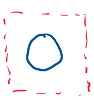


spanned by isotopy classes of framed

framed tangles ending at $\{p_1, \dots, p_n\}$.

modulo the relations

 $= v \cdot$  $+ v^{-1}$ 

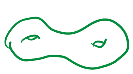
 $= (-v^2 - v^{-2})$  .

Proposition (Walker)

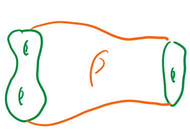
Skein modules fit into a TQFT:

$$\mathrm{Bord}^{or}_{(3,2)} \longrightarrow \mathrm{Bimod}_R$$

$$\Sigma^2 \longmapsto \mathrm{SkCat}(\Sigma)$$



$${}_{\Sigma_1}M^3{}_{\Sigma_2} \longmapsto {}_{\mathrm{SkCat}(\Sigma_1)}\mathrm{Sk\,Mod}(M)_{\mathrm{SkCat}(\Sigma_2)}$$



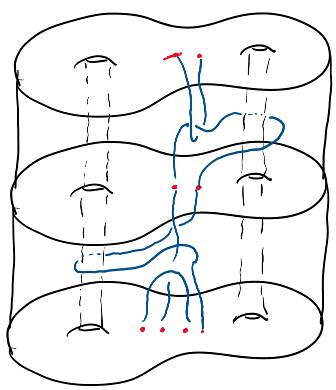
Σ a 2-manifold.

Where: $\mathrm{SkCat}(\Sigma)$ has objects given by finite subsets $\{p_1, \dots, p_k\} \subseteq \Sigma$,

$\mathrm{Hom}_{\mathrm{SkCat}(\Sigma)}\left(\{p_1, \dots, p_k\}, \{q_1, \dots, q_\ell\}\right)$ R-linear category

$$:= \mathrm{Sk}\left(\Sigma \times [0,1] \,;\, \{p_1, \dots, p_k\} \times \{0\} \cup \{q_1, \dots, q_\ell\} \times \{1\}\right)$$

Composition = "stacking"



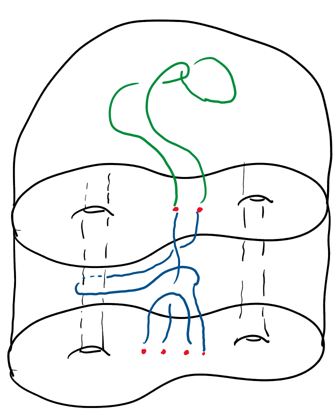
Note: $\mathrm{End}_{\mathrm{SkCat}(\Sigma)}(\emptyset) =: \mathrm{SkAlg}(\Sigma)$.

and

Given M a 3-manifold with $\partial M \cong \Sigma$

$$\mathrm{SkMod}(M) : \mathrm{SkCat}(\Sigma) \longrightarrow R\text{-mod}$$

$$\{p_1, \dots, p_k\} \longmapsto \mathrm{Sk}(M ; \{p_1, \dots, p_k\})$$



"Gluing law": $\mathrm{Sk}(M_1 \cup_\Sigma M_2) \cong \boxed{\mathrm{SkMod}(M_1) \otimes_{\boxed{\mathrm{SkCat}(\Sigma)}} \mathrm{SkMod}(M_2)}.$

- More general skein theories

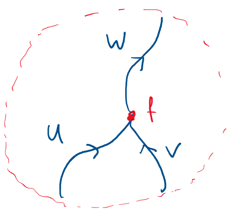
$\beta_{V,W}: V\otimes W \xrightarrow{\sim} W \otimes V$ $\quad\otimes$

Given $\mathcal{A}$ an R-linear ribbon braided monoidal category.

$\rightsquigarrow$ $\mathrm{Sk}_{\mathcal{A}}(M)$: R-module formally spanned by $\mathcal{A}$-labelled ribbon graphs, modulo local evaluation relations.

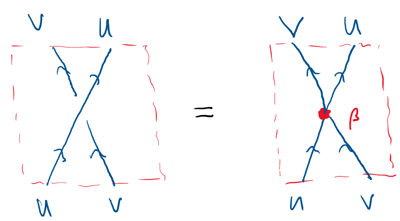
(Reshetikhin–Turaev)

$U, V, W \in \mathrm{Ob}(\mathcal{A})$



$f \in \mathrm{Hom}_{\mathcal{A}}(U \otimes V, W)$

"Universal skein relation":



Similarly, define $\mathrm{SkCat}_{\mathcal{A}}(\Sigma)$, $\mathrm{SkMod}_{\mathcal{A}}(M)$, etc.

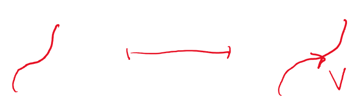
c.f. Beta factorization homology (Ayala–Francis) $\quad \mathrm{Sk}_{\mathcal{A}}(M) = H_0 \int_M \mathcal{A}$

Examples: ① $\mathcal{A} = \mathrm{Rep}_q(G)$, $R = \mathbb{C}(q^{1/d})$ $\quad$ G cx reductive group

compact Lie group

$\rightsquigarrow \mathrm{Sk}_G(M)$

e.g. $\mathrm{Sk}(M) \xrightarrow{\sim} \mathrm{Sk}_{SL_2}(M)$



V standard representation.

QCoh(BG)

② $\mathcal{A} = \mathrm{Rep}(G)$ $\quad R = \mathbb{C}$ $\quad$ $\mathrm{Map}(M_B, BG)$

AKSZ

(2) $\mathcal{A} = \mathrm{Rep}(G)$ $\quad R = \mathbb{C}$

Map$(M_B, \ldots)$

AKSZ

$\rightsquigarrow \mathrm{Sk}_{G,q=1}(M) = \mathbb{C}[\mathrm{Loc}_G(M)]$

(3) $\mathcal{A}$ a modular tensor category, $R = \mathbb{C}$

$\rightsquigarrow \mathrm{Sk}_{\mathcal{A}}(M) \cong \mathbb{C}$

framing data.

Theorem (GJS)

If M is closed then $\dim \mathrm{Sk}_{G,q}(M) < \infty$.

Langlands (Kapustin-Witten)

4d TQFT (twist of 4d $\mathcal{N}=4$ SYM G)

parameter $\Psi \in \mathbb{CP}^1 \cong S^2$

$0, \infty$

$$Z_{G,\Psi}(M) \overset{\text{proposal}}{=} H_0\, \mathrm{Sk}_{G,q}(M)$$

3-manifold

$$\boxed{q \text{ "="} e^{-\frac{1}{\Psi}}}$$

S-duality/Langlands duality: $Z_{G,\Psi} \simeq Z_{G^\vee,\Psi^\vee}$

where $G^\vee$ is the Langlands dual group

$\Psi^\vee = -1/\Psi$

hard

Conjecture: $\boxed{\dim \mathrm{Sk}_{G,q}(M) = \dim \mathrm{Sk}_{G^\vee,q^\vee}(M)}$

[We have checked this in a very small number of cases, e.g. $M = T^3$, $G = SL_2$, $G^\vee = PGL_2$.]

Conjecture: (G-Safronov) $\mathrm{Sk}_G(M) \cong H^0(\mathrm{Loc}(M); \phi)$ DT sheaf

Conjecture: (G-Safronov) $\quad \mathrm{Sk}_{G,q}(M) \cong H^0\left(\mathrm{Loc}_G(M) ; \phi\right)$ DT sheaf

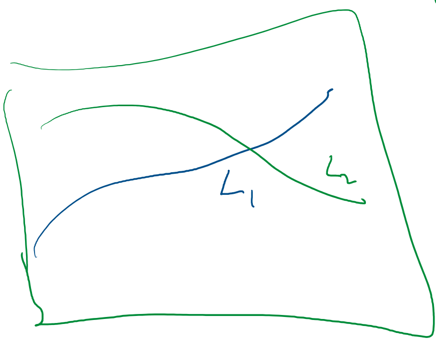
proof in progress

perverse sheaf of "vanishing cycles."

Deformation quantization $\longleftrightarrow$ DT sheaf.

Bussi-Brav-Dupont-Joyce-Szendrői

S = holomorphic symplectic manifold



$\rightsquigarrow$ W_S deformation quantization

$\circlearrowleft$

$\mathcal{M}_L$

$L_1 \cap L_2$ $\leftarrow$ (-1)-shifted symplectic

$\rightsquigarrow \exists$ perverse sheaf ϕ on $L_1 \cap L_2$ (Bussi).

$$\boxed{\begin{array}{c} R\mathrm{Hom}_{W_S}(\mathcal{M}_{L_1}, \mathcal{M}_{L_2})[?] \\ \text{or} \quad \wr\!\mathrm{S} \\ \mathcal{M}_{L_1} \overset{L}{\otimes}_{W_S} \mathcal{M}_{L_2} \end{array}} \xleftarrow{\sim} H^*(L_1 \cap L_2 ; \phi)$$

DQ vector space

[G-S]

$\hbar$

$\frac{\partial}{\partial \hbar}$

M, $\partial M = \Sigma$

$\mathrm{Loc}_G(M) \longrightarrow \mathrm{Loc}_G(\Sigma)$

e.g. $M = S^3 - N(K)$, $G = SL_2$.