Gromov–Witten theory, quantum differential equations, and derived categories

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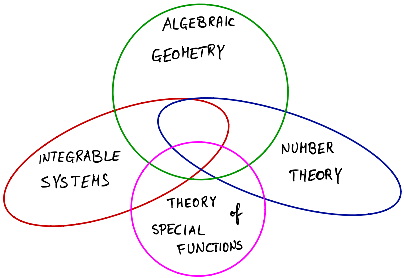
Fundação para a Ciência e a Tecnologia

Leitmotiv: To study relations between

- the topology,
- the enumerative/symplectic geometry,
- the complex geometry,

of a smooth projective variety X over $\mathbb{C}$.



Such a study is developed via the analysis of meromorphic connections on $\mathbb{P}^1(\mathbb{C})$, and their isomonodromic deformations.

X smooth projective variety $/\mathbb{C}$ $\rightsquigarrow$ $$\frac{dY}{dz} = \left(U(t) + \frac{1}{z}V(t)\right)Y$$

$t \in QH^\bullet(X)$

qDE

Why to study qDEs?

- qDEs encode enumerative geometric information
  - reconstruction and convergence of Gromov–Witten potential
- qDEs codify properties of derived categories
  - Dubrovin conjecture
- qDEs have rich number theory
  - irrationality proofs
  - distribution of prime numbers
- qDEs as a source of inspiration
  - coalescences and isomonodromic deformations
  - Borel–Laplace multi-transforms

«A proof that Euler missed» (N. Katz)

$$n^3 u_n - (34n^3 - 51n^2 + 27n - 5)u_{n-1} + (n-1)^3 u_{n-2} = 0,$$
$$a_0 = 1, \quad a_1 = 5, \qquad b_0 = 0, \quad b_1 = 1.$$

1. $|\zeta(3) - \frac{6b_n}{a_n}| = o(a_n^{-2})$
2. $a_n \in \mathbb{Z}$; $\mathrm{den}(b_n) \mid 12\,\mathrm{lcm}(1,2,\ldots,n)^3$
3. $a_n = O(\alpha^n)$, α biggest roots of $x^2 - 34x + 1$

$\Rightarrow$ $\zeta(3)$ is IRRATIONAL! (Apéry '79)

$$A(z) = \sum a_n z^n, \qquad B(z) = \sum b_n z^n$$

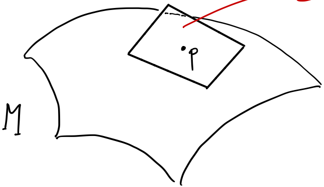
$$\mathscr{L} = \vartheta^3 - z(2\vartheta+1)(17\vartheta^2 + 17\vartheta + 5) + z^2(\vartheta+1)^3, \qquad \vartheta := z\frac{d}{dz}$$

$$\mathscr{L}A = 0, \qquad (\vartheta - 1)\mathscr{L}B = 0$$

The equation $\underline{\mathscr{L}y = 0}$ "comes from geometry", Picard-Fuchs eq (Beukers, Peters '84)

Laplace transform of qDE of Fano 3-fold V_{12} (Golyshev 07)

Frobenius Manifolds are complex manifolds whose tangent spaces admit a *Frobenius algebra* structure.



FROBENIUS ALGEBRA

$$
\begin{aligned}
x(yz) &= (xy)z, \\
xy &= yx, \\
xe &= x, \\
\eta(x, yz) &= \eta(xy, z)
\end{aligned}
$$

FLAT "METRIC"

Examples coming from:

- Symplectic and Algebraic Geometry
- Singularity Theory

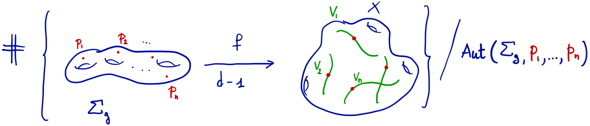
Milestones: Dubrovin, Hitchin, Kontsevich, Manin, Saito, Vafa, Witten, ...

Mirror Symmetry as isomorphism of Frobenius manifolds

$$QH^\bullet(X) \cong (V, f\colon V \to \mathbb{C})$$

Gromov-Witten theory associates with X a family of Frobenius algebras, its quantum cohomology $QH^\bullet(X)$.

$QH^\bullet(X)$ is a deformation of $H^\bullet(X)$, via counting numbers of rational curves on X



$$\langle [V_1], [V_2], \dots, [V_n] \rangle^X_{g,n,d} := \int_{[\overline{\mathcal{M}}_{g,n}(X,d)]^{virt}} ev_1^*[V_1]\, ev_2^*[V_2] \dots ev_n^*[V_n]$$

$$\overline{\mathcal{M}}_{g,n}(X,d) \xrightarrow{ev_i} X \qquad i=1,\dots,n$$

COLLECT $GW_{g=0}$ INVARIANTS INTO A GENERATING FUNCTION

$$\boxed{F_0^X(t) := \sum_{n=0}^{\infty} \sum_{d \in H_2(X,\mathbb{Z})} \sum_{\alpha_1,\dots,\alpha_n=1}^{N} \frac{t^{\alpha_1} \dots t^{\alpha_n}}{n!} \int_{[\overline{\mathcal{M}}_{0,n}(X,d)]^{virt}} \bigcup_{i=1}^{n} ev_i^* T_{\alpha_i}}$$

ASSUMPTION: F_0^X has a non-empty domain of convergence $\Omega \subseteq H^\bullet(X,\mathbb{C})$

Gromov-Witten theory associates with X a family of Frobenius algebras, its quantum cohomology $QH^\bullet(X)$.

$QH^\bullet(X)$ is a deformation of $H^\bullet(X)$, via counting numbers of rational curves on X

Two remarkable properties

1. QUASI-HOMOGENEITY

$$\boxed{\mathcal{L}_E F_0^X = (3-\dim_{\mathbb{C}} X)\, F_0^X + \text{quadratic}}$$

E Euler vector field on Ω

2. WDVV eqs

$$\boxed{\partial^3_{\alpha\beta\gamma} F \cdot \eta^{\gamma\delta} \cdot \partial^3_{\delta\varepsilon\varphi} F = \partial^3_{\varphi\beta\gamma} F \cdot \eta^{\gamma\delta} \cdot \partial^3_{\delta\varepsilon\alpha} F}$$

η Poincaré pairing

$$\eta(\alpha,\beta) := \int_X \alpha \cup \beta$$

$$c_{\alpha\beta\gamma} := \frac{\partial^3 F_X}{\partial t_\alpha \partial t_\beta \partial t_\gamma},$$

$$\frac{\partial}{\partial t_\alpha} * \frac{\partial}{\partial t_\beta} := \sum_\lambda c^\lambda_{\alpha\beta} \frac{\partial}{\partial t_\lambda}$$

COMMUTATIVITY

ASSOCIATIVITY

EXISTENCE of UNIT

&

$$\eta\left(\frac{\partial}{\partial t_\alpha} * \frac{\partial}{\partial t_\beta}, \frac{\partial}{\partial t_\gamma}\right) = \eta\left(\frac{\partial}{\partial t_\alpha}, \frac{\partial}{\partial t_\beta} * \frac{\partial}{\partial t_\gamma}\right)$$

To each point of $QH^\bullet(X)$ there is an attached differential equation

$$\frac{dY}{dz} = \left(\underline{U(t)} + \frac{1}{z}\underline{V(t)}\right) Y, \quad z \in \mathbb{C}^*, \quad t \in QH^\bullet(X).$$

$*$-MULTIPLICATION by THE EULER FIELD

$$\mathcal{U}(t) : T_t\Omega \to T_t\Omega$$
$$v \mapsto E * v$$

GRADING OPERATOR

$$V(t) : T_t\Omega \to T_t\Omega$$
$$v \mapsto \frac{2-\dim X}{2} v - \nabla_v E$$

To each point of $QH^\bullet(X)$ there is an attached differential equation

$$\frac{dY}{dz} = \left(U(t) + \frac{1}{z} V(t) \right) Y, \quad z \in \mathbb{C}^*, \quad t \in QH^\bullet(X).$$

Its solutions are multivalued, and they manifest a Stokes phenomenon.

$\to$ Monodromy data of qDE

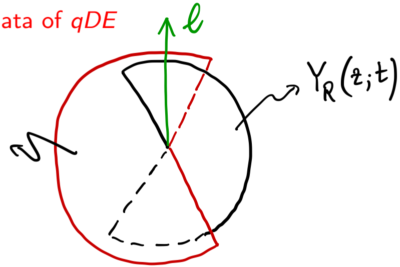
FUCHSIAN SINGULARITY $z=0$

IRREGULAR SINGULARITY $z=\infty$

$Y_0(z;t) \rightsquigarrow$ solution at $z=0$

$Y_F(z;t) \rightsquigarrow$ formal solution with exponential expansion at $z=\infty$

$Y_L(z;t)$



$$Y_R(z;t) = Y_0(z;t) \cdot C(t), \qquad Y_L(z;t) = Y_R(z;t) \cdot S(t)$$

- Isomonodromic property: monodromy data are locally constant wrt t.

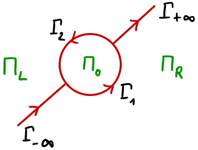
RHB inverse problem

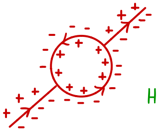
$$\begin{tikzcd} \text{Frobenius Manifold } QH^{\bullet}(X) \arrow[r, bend left] & \text{Monodromy Data } (\mu, R, S, C) \arrow[l, bend left, "\text{RHB}"] \end{tikzcd}$$

The monodromy data define a "system of coordinates" in the space of quasi-homogeneous solutions of WDVV equations

RHB inverse problem

Let $u_0 \in \mathbb{C}^n \setminus \Delta$ inside an ℓ-chamber





$H : \Gamma \to GL(n, \mathbb{C})$

$$H(z; u_0) = \begin{cases} e^{zU}\, S\, e^{-zU} & z \in \Gamma_{+\infty} \\ e^{zU}\, S^T\, e^{-zU} & z \in \Gamma_{-\infty} \\ e^{zU}\, C^{-1} z^{-R} z^{-\mu} & z \in \Gamma_1 \\ S^{-1}\, C^{-1} z^{-R} z^{-\mu} & z \in \Gamma_2 \end{cases}$$

RHB Problem Find an analytic $G : \mathbb{C} \setminus \Gamma \to M_n(\mathbb{C})$ such that

1. $G|_{\Pi_\nu} \in \mathcal{C}^0(\overline{\Pi_\nu})$, $\nu = 0, L, R$
2. $\exists$ non-tangential limits $G_\pm : \Gamma^\circ \to M_n(\mathbb{C})$ and $G_+ = G_- H$
3. $G(z) \to I_n$ as $z \to \infty$

Thm (Malgrange) If the RHB problem is solvable at $u_0 \in \mathbb{C}^n \setminus \Delta$, then it is solvable in a neighborhood of u_0. The solution in holomorphic on u.

If $G(z, u) = G_0(u) + z G_1(u) + z^2 G_2(u) + z^3 G_3(u) + O(z^4)$, $\quad z \to 0$ in Π_0

then

$$t^\alpha(\boldsymbol{u}) := \eta^{\alpha\beta} \sum_{i=1}^n G_{0,i\beta}(\boldsymbol{u}) G_{1,i1}(\boldsymbol{u}), \quad \alpha = 1, \dots, n,$$

$$F(\boldsymbol{u}) := \frac{1}{2}\left[t^\alpha(\boldsymbol{u}) t^\beta(\boldsymbol{u}) \sum_{i=1}^n G_{0,i\alpha}(\boldsymbol{u}) G_{1,i\beta}(\boldsymbol{u}) - \sum_{i=1}^n G_{1,i1}(\boldsymbol{u}) G_{2,i1}(\boldsymbol{u}) G_{0,i1}(\boldsymbol{u}) G_{3,i1}(\boldsymbol{u}) \right].$$

This is a solution of WDVV

1. Reconstruction of WDVV potential:

Malgrange Theorem ('80s)

on the existence of solutions of families of RHB problems

2. Extension of Malgrange Theorem: C. Sabbah (2021)

based on results of T. Mochizuki

3. Convergence Theorem (C. 2021)

Let $F \in \mathbb{C}[[t^1, \dots, t^n]]$, quasi-homogeneous, WDVV-potential.

If F defines a semisimple Frobenius algebra at $t=0$,

then $F \in \mathbb{C}\{t^1, \dots, t^n\}$.

Refined in [CDG18]

Motivation: tt^*-geometry

Dubrovin Conjecture

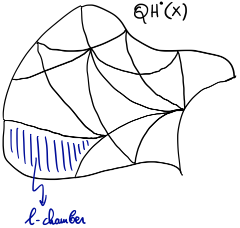
$$\text{Symplectic and Enumerative Geometry of } X\text{: } QH^\bullet(X)$$
$$\Big\downarrow{\scriptstyle qDE}$$
$$\text{Complex geometry of } X\text{: } \mathcal{D}^b(X)$$

The monodromy data of the *qDE* of X are determined by

- the topology of X (dimension, characteristic classes),
- characteristic classes of exceptional collections in $\mathcal{D}^b(X)$

$$(E_i)_{i=1}^n, \quad \mathrm{Hom}^\bullet(E_i, E_i) \cong \mathbb{C}, \quad \mathrm{Hom}^\bullet(E_j, E_i) = 0, \quad j > i.$$

$$S^{-1} = \left(\chi(E_i, E_j)\right)_{ij}, \quad C = \frac{(\sqrt{-1})^d}{(2\pi)^{\frac{d}{2}}} \hat{\Gamma}_X^- \cup e^{-\pi\sqrt{-1}c_1(X)} \cup Ch(E_j)$$



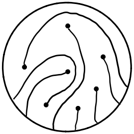
TO EACH CHAMBER CORRESPONDS an EXCEPTIONAL COLLECTION

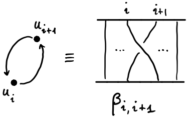
$$(S^{-1})_{ij} = \chi(E_i, E_j)\,,$$

$$C_k = \underbrace{\frac{1}{(2\pi\sqrt{-1})^{\frac{\dim X}{2}}}\,\hat{\Gamma}_X^-\, e^{-\pi\sqrt{-1}\,c_1(X)}}_{\text{topological invariant of } X}\, Ch(E_k)$$

WALL-CROSSING PHENOMENON:

BRAID GROUP B_n - ACTION on BOTH (S,C) AND EXC. COLL.



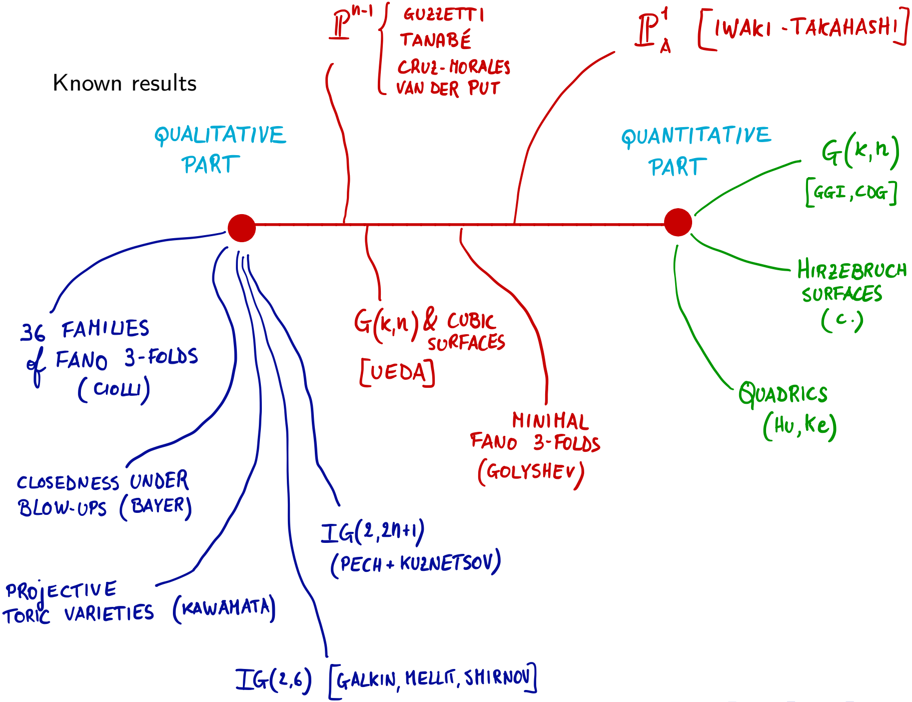


DICTIONARY

$$\left\{\begin{array}{c}\text{Analysis of}\\ \text{iso-def. of qDE}\end{array}\right\} \rightleftarrows \left\{\begin{array}{c}\text{Geometry of}\\ \mathcal{D}^b(X)\end{array}\right\}$$

IMPORTANT REMARK: $\mathcal{D}^b(X) \rightsquigarrow$ Riemann-Hilbert-Birkhoff boundary value problem $\rightsquigarrow$ Reconstruction of GW-theory of X

Known results



Cotti, Varchenko. In *Integrability, Quantization, and Geometry*, AMS, 2020

Equivariant framework: let G act on X.

$$QH^\bullet_G(X) \xrightarrow[qDE+qKZ]{\quad\cdots\cdots\quad} \mathcal{D}^b_G(X)$$

Maulik, Okounkov, Tarasov, Varchenko: the qDE admits a compatible system of difference equations

$$Y(t, z_1, \dots, z_i - 1, \dots, z_m) = K_i(t, z)Y(t, z), \quad i = 1, \dots, m.$$

$$\begin{tikzcd} \text{Equivariant K-theory} \arrow[rr, "\text{Б}"] \arrow[dr, bend right, "TV"'] & & \text{Equivariant Cohomology} \arrow[dl, bend left, "S_o"] \\ & \text{Solutions of qDE} & \end{tikzcd}$$

$$\text{Б}(E) := e^{\pi\sqrt{-1}\,c_1(X)}\,\hat{\Gamma}^+_X\,Ch(E)$$

$S_o(\alpha)$ is a suitable generating function of GW-invariants with gravitational descendants

Explicit computations for $X = \mathbb{P}^{n-1}$ and $G = \mathbb{T}^n$.

Results: perfect equivariant lifts of my previous results!

Coalescence problem for isomonodromic deformations

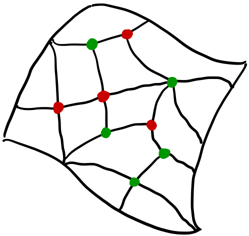
Cotti, Dubrovin, Guzzetti. Duke Math. Journal, Volume 168, Number 6 (2019), 967-1108.

Cotti, Dubrovin, Guzzetti. SIGMA 16 (2020), 040, 105 pages.

Theory of non-generic Isomonodromic Deformations

$$\frac{dY}{dz} = \left(U(t) + \frac{1}{z} V(t) \right) Y, \quad U = \mathrm{diag}(u_1(t), \dots, u_n(t))$$

- Main problem: extend the analytical theory when $u_i(t) = u_j(t)$, $i \neq j$.
- Results: Formal solutions, Asymptotics, Stokes phenomenon, (isomonodromic) deformation theory...
- Applications: Frobenius manifolds, Painlevé transcendents, Riemann-Hilbert problems...



- COALESCING POINTS with SEMISIMPLE FROB. ALGEBRA
- COALESCING POINTS with NON-SEMISIMPLE FROB. ALG.

IT MAY HAPPEN THAT THE FROBENIUS STRUCTURE is KNOWN ONLY AT ● POINTS !!

FRAMEWORK of CDG-Results

GENERAL DEFORMATIONS THEORY BOTH ISOMONODROMIC AND NOT for SYSTEMS $\frac{dY}{dz} = \left(U(t) + \frac{1}{z}V(t)\right)Y$

PHILOSOPHY of MAIN CDG-MAIN RESULTS :

To find sharp conditions on the coeff.'s of the def. so that the analysis is TAME

FROBENIUS MANIFOLDS CASE:

All sharp conditions are surprisingly implied by the axioms defining the Frobenius Structure !!!

The analysis at ●-points is well-behaved!

Natural questions : is it really so interesting ??

How much often coalescences arise ?

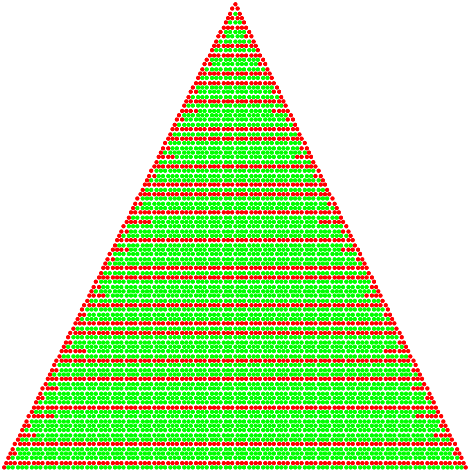


Figure: Surprising connection with prime number distribution: equivalent formulations of RH. Cotti. *IMRN*, doi: 10.1093/imrn/rnaa163, 2020

$G(k,n)$ is COALESCING iff $\pi_1(n) \leq k \leq n - \pi_1(n)$.

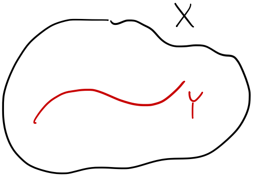
QUANTUM SATAKE PRINCIPLE

Integral representations of solutions of *qDE*

Problem: How to find bases of solutions of the *qDE*?

CASE I

$Y \subseteq X$

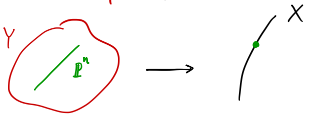


$Y = \{x \in X : \sigma(x) = 0\}$

where $\sigma : X \to E$ is a section of a vector bundle $E \to X$.

CASE II

$Y \to X$



$Y = \mathbb{P}(V)$

where $V \to X$ vector bundle

Question: Assume we "know" how to solve the qDE of X. Can we solve the qDE of Y?

Positive Answers, under some splitting assumption

In CASE I :

- E is a direct sum of fractional powers of $\det TX$
- $X = X_1 \times \dots \times X_k$ is a product of Fano varieties, and E is the external tensor product of fractional powers of the determinant bundles $\det TX_i$

In CASE II :

- $X = X_1 \times \dots \times X_k$ is a product of Fano varieties, and V is

$$V \simeq \mathcal{O}_X \oplus L$$

where L is the external tensor product of fractional powers of the determinant bundles $\det TX_i$

In both cases, both X and Y are assumed to be Fano.

$\Rightarrow$ Reduction to a scalar *qDE*

MASTER FUNCTIONS

The qDE can be equivalently written for a scalar function

$$\boxed{\Phi(z,t) := z^{-\frac{\dim X}{2}} \int_X Y(z,t)}$$

Theorem (C. to appear in MEMS)

For generic $t \in QH^\bullet(X)$, the solution $Y(z,t)$ can be reconstructed from $\Phi(z,t)$

RK: This is equivalent to the choice of the unit vector field e as a cyclic vector

Examples:

- projective space $\mathbb{P}^{n-1}$: $\vartheta^n \Phi = (nz)^n \Phi, \quad \vartheta := z\frac{d}{dz}$,
- $G(2,4)$: $\vartheta^5\Phi - 1024z^4\vartheta\Phi - 2048z^4\Phi = 0$,
- $\widetilde{\mathbb{P}^2}$:

$$(283z-24)\vartheta^4\Phi + \left(283z^2 - 590z + 24\right)\vartheta^3\Phi + \left(-2264z^2 + 192z + 3\right)\vartheta^2\Phi$$
$$-4z^2\left(2547z^2 + 350z - 104\right)\vartheta\Phi + z^2\left(-3113z^3 - 9924z^2 + 1476z + 192\right)\Phi = 0.$$

CASE I : $Y \subseteq X$

Theorem (C., *MEMS*)

Assume

$L \to X$ ample line bundle, $\quad E = \bigoplus_{j=1}^{r} L^{\otimes d_j}, \quad \det TX = L^{\otimes \ell}, \quad \ell > \sum_{j=1}^{r} d_j$

then there exists $c \in \mathbb{C}$ such that any master function of Y is a $\mathbb{C}$-linear combination of integrals of the form

$$e^{-cz}\mathscr{L}_{\frac{\ell-\sum_{i=1}^{s} d_i}{d_s}, \frac{d_s}{\ell-\sum_{i=1}^{s-1} d_i}} \circ \cdots \circ \mathscr{L}_{\frac{\ell-d_1-d_2}{d_2}, \frac{d_2}{\ell-d_1}} \circ \mathscr{L}_{\frac{\ell-d_1}{d_1}, \frac{d_1}{\ell}}[\Phi]$$

$$= e^{-cz}\int_0^\infty \cdots \int_0^\infty \Phi\left(z^{\frac{\ell-\sum_{j=1}^{r} d_j}{\ell}} \prod_{i=1}^{r} \zeta_i^{\frac{d_i}{\ell}}\right) e^{-\sum_{i=1}^{r}\zeta_i} d\zeta_1 \ldots d\zeta_r,$$

where Φ is a master function of X.

CASE I : $Y \subseteq X$

Theorem (C., *MEMS*)

Assume

$X = \prod_{j=1}^{h} X_j$, $\quad L_j \to X_j$ ample line bundle, $\quad \det TX_j = L_j^{\otimes \ell_j}$, $\quad E = \boxtimes_{j=1}^{h} L_j^{\otimes d_j}$, $\quad \ell_j > d_j$

then there exists $c \in \mathbb{C}$ *such that any master function of* Y *is a* $\mathbb{C}$*-linear combination of integrals of the form*

$$e^{-cz}\mathscr{L}_{\alpha,\beta}[\Phi_1,\dots,\Phi_h](z) = e^{-cz}\int_0^{\infty}\prod_{j=1}^{h}\Phi_j\left(z^{\frac{\ell_j-d_j}{\ell_j}}\lambda^{\frac{d_j}{\ell_j}}\right)e^{-\lambda}d\lambda,$$

where $(\alpha,\beta) = \left(\frac{\ell_1-d_1}{d_1},\dots,\frac{\ell_h-d_h}{d_h};\frac{d_1}{\ell_1},\dots,\frac{d_h}{\ell_h}\right)$, Φ_j *is a master function of* X_j.

CASE II : $Y = \mathbb{P}(V)$, $V \to X$

Theorem (C., arXiv:2210.05445, to appear in *JMPA*) $[\ell_j > d_j]$

Assume

$X = \prod_{j=1}^{h} X_j$, $L_j \to X_j$ ample line bundle, $\det TX_j = L_j^{\otimes \ell_j}$, $L = \boxtimes_{j=1}^{h} L_j^{\otimes d_j}$, $V = \mathcal{O}_X \oplus L$,

then any master function of Y is a $\mathbb{C}$-linear combination of integrals of the form

$$\mathscr{B}_{\alpha,\beta}[\Phi_1, \dots, \Phi_h, \mathcal{E}_k], \quad k = 0, \dots, 1 + \dim_{\mathbb{C}} X,$$

where

$$\alpha = \left(\frac{\ell_1^2}{d_1(d_1 - \ell_1)}, \dots, \frac{\ell_h^2}{d_h(d_h - \ell_h)}, \frac{1}{2} \right), \qquad \beta = \left(-\frac{d_1}{\ell_1}, \dots, -\frac{d_h}{\ell_h}, 1 \right),$$

that is of the form

$$\frac{1}{2\pi\sqrt{-1}} \int_{\mathfrak{H}} \prod_{j=1}^{h} \Phi_j \left(z^{\frac{\ell_j - d_j}{\ell_j}} \lambda^{\frac{d_j}{\ell_j}} \right) \mathcal{E}_k \left(z^2 \lambda^{-1} \right) e^{\lambda} \frac{d\lambda}{\lambda},$$

with Φ_j master function of X_j.

$\mathcal{E}(s,z) := e^z z^s \gamma^*(s,z), \qquad \mathcal{E}_k(z) := \left.\frac{\partial^k}{\partial s^k} \mathcal{E}(s,z)\right|_{s=0}.$

$\gamma^*(s,z) := \frac{z^{-s}}{\Gamma(s)} \gamma(s,z)$

An example : $Bl_{pt}\mathbb{P}^2$

$$Bl_{pt}\mathbb{P}^2 \cong \begin{cases} \text{hypersurface in} \\ \mathbb{P}^1\times\mathbb{P}^2 \text{ of bidegree } (1,1) \\ \mathbb{P}(\mathcal{O}\oplus\mathcal{O}(-1)) \to \mathbb{P}^1 \end{cases}$$

qDE of $Bl_{pt}\mathbb{P}^2$:

$$(283z-24)\vartheta^4\Phi + (283z^2-590z+24)\,\vartheta^3\Phi + (-2264z^2+192z+3)\,\vartheta^2\Phi$$
$$-4z^2(2547z^2+350z-104)\,\vartheta\Phi + z^2(-3113z^3-9924z^2+1476z+192)\,\Phi = 0.$$

Two possible ways to integrate this equation

$$S(\mathbb{P}^1) = \{\Phi_1 : \vartheta^2\Phi_1 = 4z^2\Phi_1\}$$

$$\Phi_1(z) = \sum_{m=0}^{\infty}\left(A_{m,1} + A_{m,0}\log z\right)\frac{z^{2m}}{(m!)^2}$$

$$S(\mathbb{P}^2) = \{\Phi_2 : \vartheta^3\Phi_2 = 27z^3\Phi_2\}$$

$$\Phi_2(z) = \sum_{n=0}^{\infty}\left(B_{n,2} + B_{n,1}\log z + B_{n,0}\log^2 z\right)\frac{z^{3n}}{(n!)^3}$$

$$\mathscr{P}: \mathcal{S}(\mathbb{P}^1)\otimes\mathcal{S}(\mathbb{P}^2) \to \mathcal{O}(\widetilde{\mathbb{C}^*})$$

$$\mathscr{P}(\Phi_1\otimes\Phi_2;\ z) := e^{-z}\mathscr{L}_{(1,2;\frac{1}{2},\frac{1}{3})}[\Phi_1,\Phi_2;\ z]$$
$$= e^{-z}\int_0^{\infty}\Phi_1\left(z^{\frac{1}{2}}\lambda^{\frac{1}{2}}\right)\Phi_2\left(z^{\frac{2}{3}}\lambda^{\frac{1}{3}}\right)e^{-\lambda}\,d\lambda,$$

Let $\mathcal{H} \subseteq S(\mathbb{P}^1)\otimes S(\mathbb{P}^2)$

$$\mathcal{H}: \begin{cases} A_{00}B_{00} = 0 \\ 4A_{0,1}B_{0,0} = 3A_{0,0}B_{0,1} \end{cases}$$

$\mathscr{P}(\mathcal{H})$ is the space of solution of the qDE

An example : $Bl_{pt}\mathbb{P}^2$

$Bl_{pt}\mathbb{P}^2 \cong \begin{cases} \text{hypersurface in } \mathbb{P}^1\times\mathbb{P}^2 \text{ of bidegree } (1,1) \\ \mathbb{P}(\mathcal{O}\oplus\mathcal{O}(-1)) \to \mathbb{P}^1 \end{cases}$

qDE of $Bl_{pt}\mathbb{P}^2$:

$$(283z-24)\vartheta^4\Phi + (283z^2-590z+24)\,\vartheta^3\Phi + (-2264z^2+192z+3)\,\vartheta^2\Phi$$
$$-4z^2(2547z^2+350z-104)\,\vartheta\Phi + z^2(-3113z^3-9924z^2+1476z+192)\,\Phi = 0.$$

Two possible ways to integrate this equation

$$S(\mathbb{P}^1)=\left\{\Phi_1 : \ \vartheta^2\Phi_1 = 4z^2\Phi_1\right\}$$

$$\Phi_1(z)=\sum_{m=0}^{\infty}\left(A_{m,1}+A_{m,0}\log z\right)\frac{z^{2m}}{(m!)^2}$$

$\Phi_1^{(1)} \rightsquigarrow A_{m,0}=0, \quad A_{m,1}=1$

$\Phi_1^{(2)} \rightsquigarrow A_{m,0}=1, \quad A_{m,1}=0$

The functions

$$\mathcal{B}_{\alpha,\beta}\left[\Phi_1^{(1)},\mathcal{E}_0\right] \qquad \mathcal{B}_{\alpha,\beta}\left[\Phi_1^{(1)},\mathcal{E}_1\right] \qquad \mathcal{B}_{\alpha,\beta}\left[\Phi_1^{(2)},\mathcal{E}_0\right]$$

$$4\mathcal{B}_{\alpha,\beta}\left[\Phi_1^{(2)},\mathcal{E}_1\right]+\mathcal{B}_{\alpha,\beta}\left[\Phi_1^{(1)},\mathcal{E}_2\right]$$

$\alpha=\left(-4,\frac{1}{2}\right)$

$\beta=\left(-\frac{1}{2},1\right)$

span the space of solutions of the qDE.

$\Phi_1 \in S(\mathbb{P}^1)$

$$\Phi_1(z) = \sum_{m=0}^{\infty} \left(A_{m,1} + A_{m,0} \log z\right) \frac{z^{2m}}{(m!)^2}$$

$\Phi_2 \in S(\mathbb{P}^2)$

$$\Phi_2(z) = \sum_{n=0}^{\infty} \left(B_{n,2} + B_{n,1} \log z + B_{n,2} \log^2 z\right) \frac{z^{3n}}{(n!)^3}$$

$\rightarrow$ $(A_{0,i}\, B_{0,j})$ with $i=0,1$ and $j=0,1,2$ are coordinates on $S(\mathbb{P}^1) \otimes S(\mathbb{P}^2)$.

$$\mathcal{H}: \begin{cases} A_{00} B_{00} = 0 \\ 4 A_{0,1} B_{0,0} = 3 A_{0,0} B_{0,1} \end{cases}$$

$\mathcal{H}$ is iso to solutions of qDE of $\mathbb{F}_1$

$\rightarrow$ Reconstruction of Stokes bases of solutions

The central connection matrix of $\mathbb{F}_{2k+1}$ is

$$C_k = \begin{pmatrix} \frac{1}{2\pi} & -\frac{1}{2\pi} & \frac{1}{2\pi} & -\frac{1}{2\pi} \\ \frac{\gamma}{\pi} & -\frac{\gamma}{\pi} & i + \frac{\gamma}{\pi} & -i - \frac{\gamma}{\pi} \\ \frac{\gamma - 2\gamma k - i\pi}{2\pi} & -\frac{\gamma - 2\gamma k + i\pi}{2\pi} & \frac{-2\gamma k - i(2\pi k + \pi) + \gamma}{2\pi} & \frac{(2k-1)(\gamma + i\pi)}{2\pi} \\ \gamma\left(-i + \frac{2\gamma}{\pi}\right) & \gamma\left(-i - \frac{2\gamma}{\pi}\right) & \frac{2\gamma(\gamma + i\pi)}{\pi} & -\frac{2(\gamma + i\pi)^2}{\pi} \end{pmatrix}.$$

Theorem

Dubrovin conjecture holds true for all Hirzebruch surfaces.

The matrix C_k is the matrix associated with the morphism

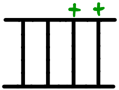
$$Д^{-}_{\mathbb{F}_{2k+1}} : K_0(\mathbb{F}_{2k+1})_{\mathbb{C}} \to H^{\bullet}(\mathbb{F}_{2k+1}, \mathbb{C}), \quad [\mathscr{F}] \mapsto \frac{1}{2\pi} \widehat{\Gamma}^{-}_{\mathbb{F}_{2k+1}} \cup e^{-\pi i c_1(\mathbb{F}_{2k+1})} \cup \mathrm{Ch}(\mathscr{F}),$$

w.r.t. an exceptional basis $\mathfrak{E} := (E_i)_{i=1}^{4}$ of $K_0(\mathbb{F}_{2k+1})_{\mathbb{C}}$.

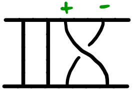
$\mathbb{F}_k \equiv \mathcal{O}(-k) \cup \infty$ section

fiber of $\mathcal{O}(-k)$ ∞-section

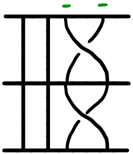
The exceptional collection $\mathfrak{E}$ is obtained from $(\mathcal{O}, \mathcal{O}(\Sigma_2), \mathcal{O}(\Sigma_4), \mathcal{O}(\Sigma_2 + \Sigma_4))$ by applying the following elements of $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathcal{B}_4$:



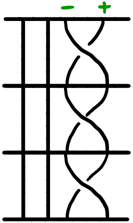
$\mathbb{F}_1$



$\mathbb{F}_3$



$\mathbb{F}_5$



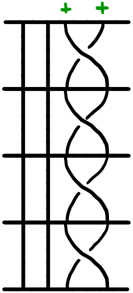
$\mathbb{F}_7$

Different Complex Structures

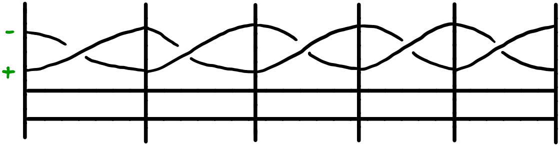
$\Downarrow$

increasing powers of the SAME BRAID

β_{34}^k



$\mathbb{F}_9$



$\mathbb{F}_{11}$

OBRIGADO!