Rigidity of non-compact static domains in hyperbolic space via positive mass theorems

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Outline of the talk

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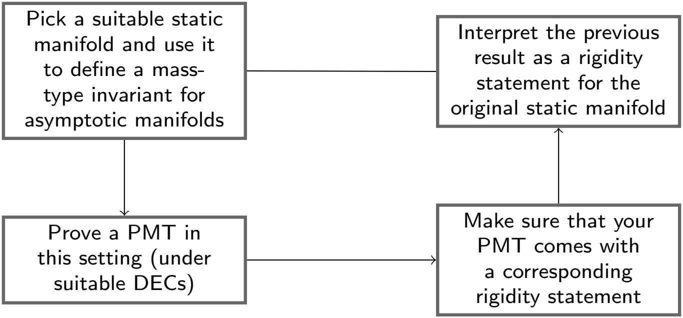

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$$|g - \delta|_\delta = O_2(r^{-\sigma}), \quad \sigma > n - 2,$$

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[1] R. Souam, Mean curvature rigidity of horospheres, hyperspheres, and hyperplanes, *Archiv der Mathematik*, 2021.

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- We consider a manifold $\widetilde{M}^{n+1}$ endowed with a boundary $\partial\widetilde{M}$. On the space of all Lorentzian metrics $\widetilde{g}$ on $\widetilde{M}$ such that $\partial\widetilde{M}$ is time-like, we may consider the Gibbons-Hawking-York functional

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We say that a bordered Riemannian manifold $(M, g, \partial M)$ is a static manifold with boundary, with the pair $(\widetilde{\Lambda}, \widetilde{\lambda})$ as cosmological constants, if there exists $V \neq 0$ such that the equations above are satisfied. In this case, each such V is termed a static potential.

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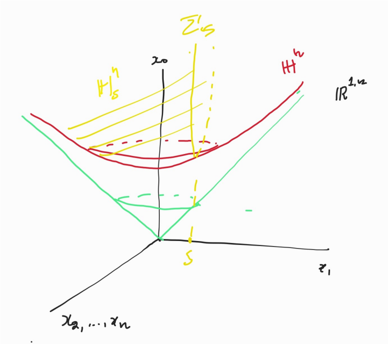

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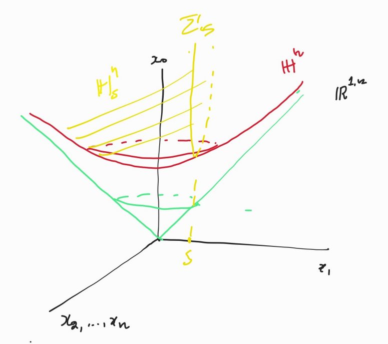

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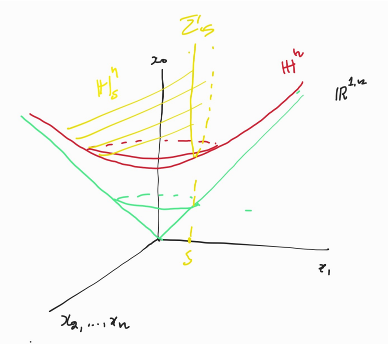

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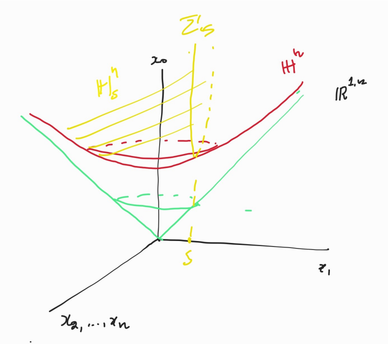

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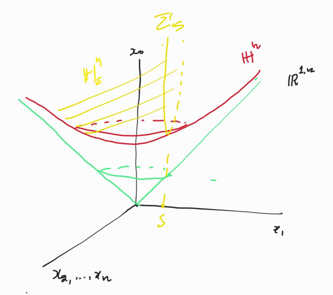

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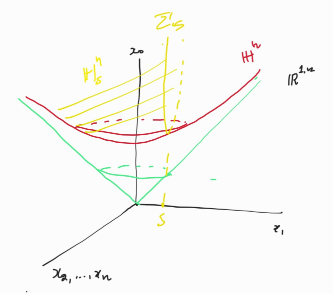

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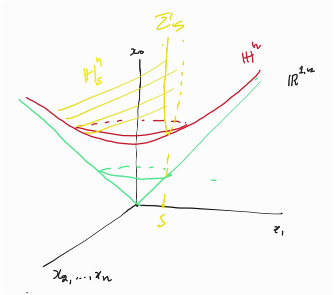

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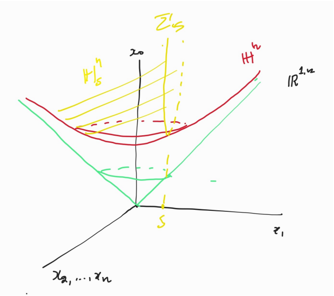

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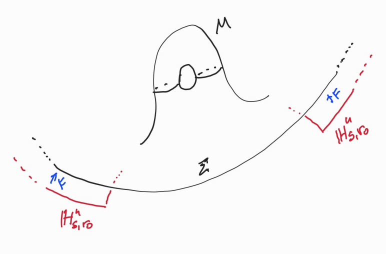

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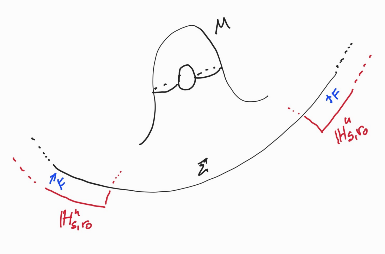

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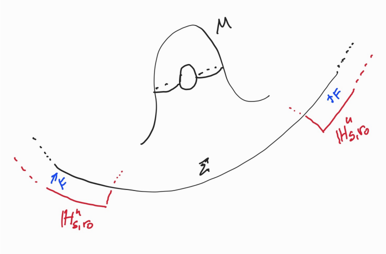

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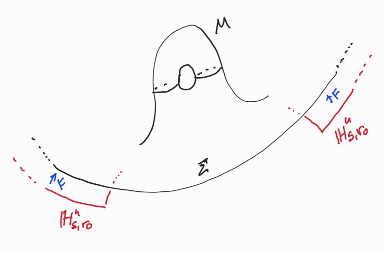

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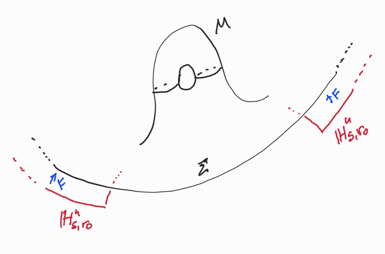

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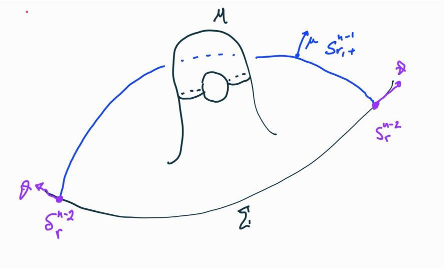

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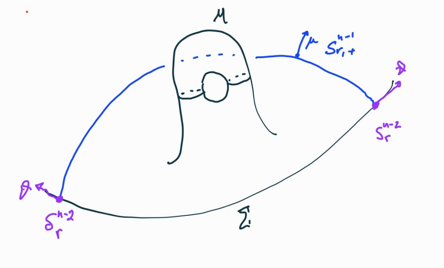

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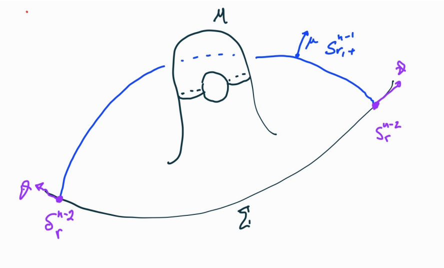

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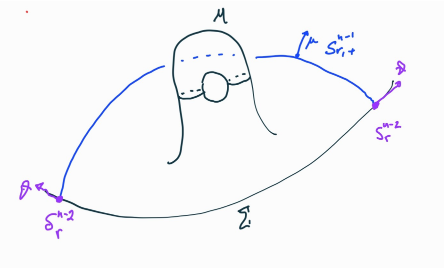

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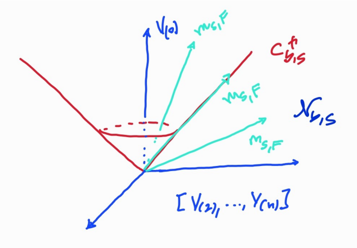
It follows that ρ^s acts isometrically on $(\mathcal{N}_{b,s}, \langle\,,\rangle^s_{n-1,1})$. In particular, the causal properties of $\mathfrak{m}_{s,F}$ are chart independent.



Invariance of the mass vector

- If F_1 and F_2 are charts at infinity then $F_{12} := F_1^{-1} \circ F_2 : \mathbb{H}^n_s \to \mathbb{H}^n_s$ satisfies $F_{12}^* b = b + O_2(r^{-\sigma})$. Because $\mathbb{H}^n_s$ is ''rigid'' at infinity, there exists $A \in O^{\uparrow}(1, n-1)$ such that $F_{12} = A + O_2(r^{-\sigma})$

Proposition (equivariance of the mass)

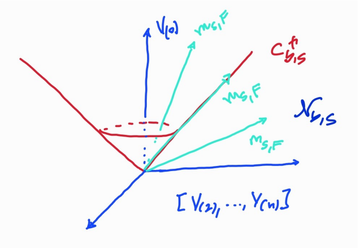
Under the conditions above,

$$\mathfrak{m}_{s,F_1} = \rho^{s*}_{A^{-1}}(\mathfrak{m}_{s,F_2}),$$

where ρ^{s*} is the dual representation.

- We may identify $\mathcal{N}_{b,s} \equiv \mathbb{R}^{n-1,1}$ by introducing the '"Lorentzian'' metric $\langle\, , \rangle^s_{n-1,1}$ and declaring that $V_{(0)}$ is ''time-like'' (in the sense that $\langle V_{(0)}, V_{(0)}\rangle^s_{n-1,1} = 1$) and $\langle V_{(a)}, V_{(b)}\rangle^s_{n-1,1} = -\delta_{a,b}$, $2 \le a, b \le n$.

It follows that ρ^s acts isometrically on $(\mathcal{N}_{b,s}, \langle\, , \rangle^s_{n-1,1})$. In particular, the causal properties of $\mathfrak{m}_{s,F}$ are chart independent.



The positive mass theorem

Theorem (Almaraz, —, 2022)

Let (M, g, Σ) be an s-AH spin manifold with $R_g \geq -n(n-1)$ and $H_g \geq (n-1)\lambda_s$. Then, for any chart F as above, the mass vector $\mathfrak{m}_{s,F}$ is time-like and future directed unless it vanishes, in which case (M, g, Σ) is isometric to $(\mathbb{H}^n_s, b, \Sigma_s)$.

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- The terminology is justified by the fact that the numerical invariant

$$\mathfrak{m}_s := \sqrt{\langle \mathfrak{m}_{s,F}, \mathfrak{m}_{s,F} \rangle^s_{1,n}},$$

does not depend on the chosen chart and may be regarded as the total mass of the isolated gravitational system whose (time-symmetric) initial data set is (M, g, Σ). Hence, $\mathfrak{m}_s \geq 0$ with the equality holding if and only if (M, g, Σ) is isometric to $(\mathbb{H}^n_s, b, \Sigma_s)$.

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- Assume that M is a spin manifold with a fixed spin structure. In the presence of a metric g, there exists a canonical hermitean vector bundle $\mathbb{S}M \to M$, the spinor bundle of (M, g), endowed with a compatible connection ∇. Elements of $\Gamma(\mathbb{S}M)$ are called spinors.

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- Define the Killing connection by

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where $\mathfrak{c}$ is Clifford multiplication, and the corresponding Killing-Dirac operators by $D^{\pm} : \Gamma(\mathbb{S}M) \to \Gamma(\mathbb{S}M)$ by the composition

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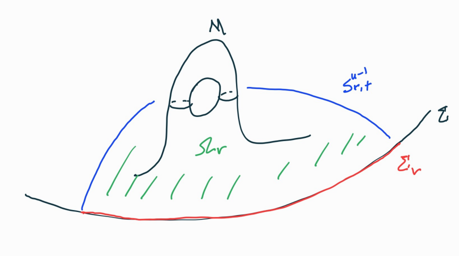

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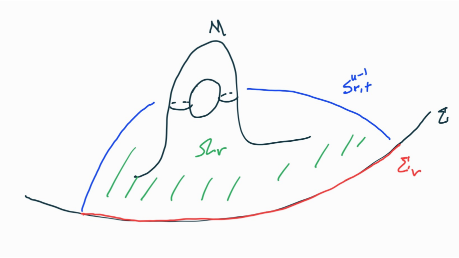

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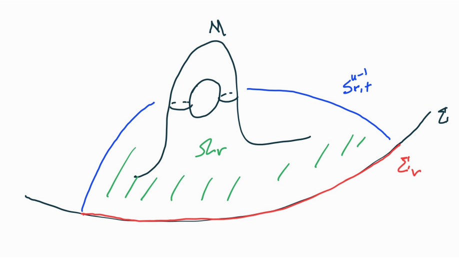

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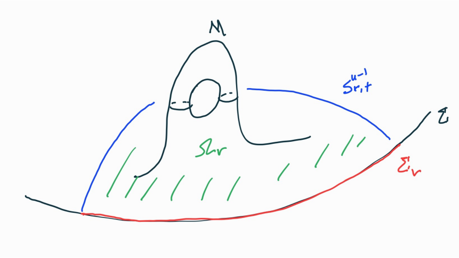

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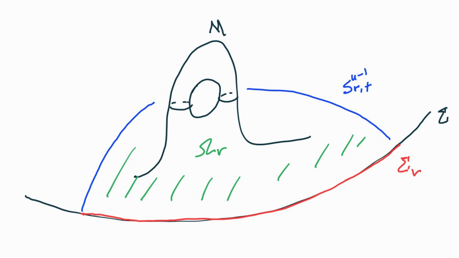

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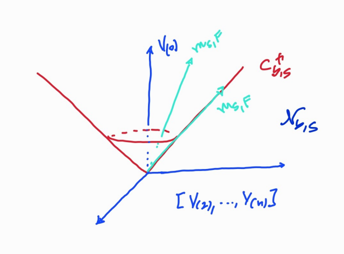
The DECS then imply that $\langle \mathfrak{m}_{s,F}, V\rangle_{n,1} \geq 0$, for any $V \in \mathcal{C}^{\uparrow}_{b,s}$, which means that $\mathfrak{m}_{s,F}$ is is time-like unless there exists a Killing spinor $\Psi^\theta \not\equiv 0$ on M meeting the corresponding θ-boundary condition along Σ.



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How the mass formula implies our main result I

- Since any $V \in \mathcal{C}^{\uparrow}_{b,s}$ is of the form $V = V_\Phi$ for some Killing spinor Φ on $\mathbb{H}^n_s$,

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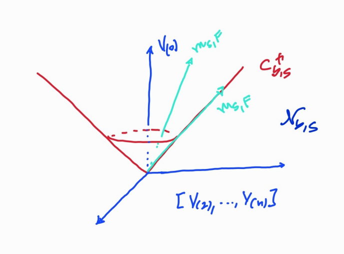

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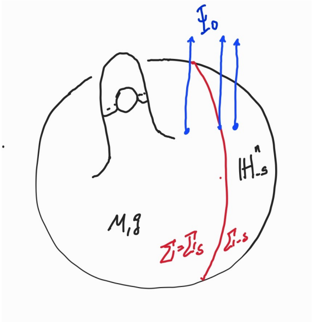
This allows us to glue (M, g, Σ) to $(\mathbb{H}^n_{-s}, b, \Sigma_{-s})$ along the common boundary $\Sigma = \Sigma_{-s}$ to obtain a boundaryless n-manifold which is AH (with $\mathbb{H}^n$ as its model at infinity), Einstein and carries a Killing spinor Ψ^θ. We conclude that this glued manifold is isometric to $(\mathbb{H}^n, b)$ and hence (M, g, Σ) is isometric to $(\mathbb{H}^n_s, b, \Sigma_s)$, as desired.



The horospherical case

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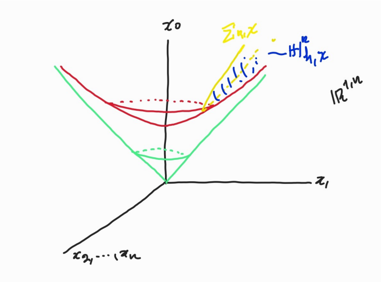
By means of the hyperboloid model $\mathbb{H}^n \hookrightarrow \mathbb{R}^{1,n}$, we consider the horoball

$$\mathbb{H}^n_h = \{x \in \mathbb{H}^n; x_0 - x_1 \leq 1\}.$$

We denote by Σ_h its boundary.



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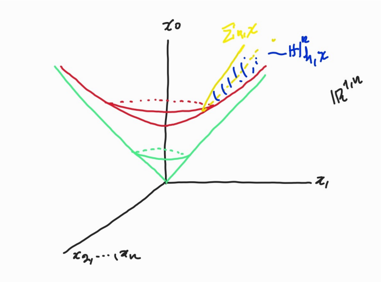

Proposition

$(\mathbb{H}^n_h, b, \Sigma_h)$ is a static domain whose boundary Σ_h is a horosphere (with mean curvature $n-1$). In this case, $(\widetilde{\Lambda}, \widetilde{\lambda}) = (-n(n-1)/2, n-1)$ and the corresponding space of static potentials is

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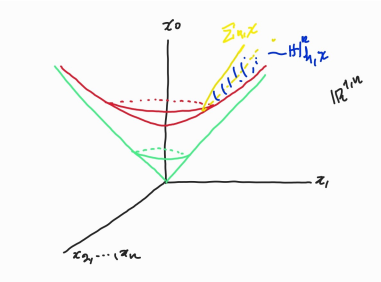

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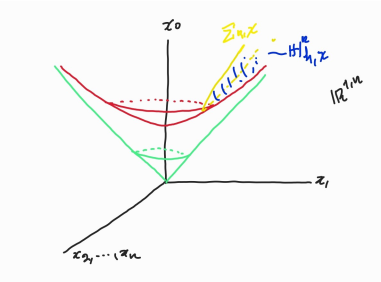

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