Topological defects form higher dagger categories

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based on joint work with: Bruce Bartlett, Gio Ferrer, Brett Hungar, Theo Johnson-Freyd, Cameron Krulewski, Nivedita, Dave Penneys, David Reutter, Claudia Scheimbauer, Luuk Stehouwer, Chetan Vuppulury

Def.: A $\dagger$-category is a category $\mathcal{C}$ together with a functor $\dagger: \mathcal{C} \to \mathcal{C}^{op}$, s.t. $\dagger(c) = c$ & $\dagger^2 = id_{\mathcal{C}}$.

Examples: • Hilb, bounded operators, $\dagger =$ adjoint.

• $\mathrm{Bord}_n^{SO(n)}$, $(M: \Sigma \longrightarrow \Sigma')^\dagger = \overline{M}: \Sigma' \longrightarrow \Sigma$

orientation reversal

Rem: This definition is "evil", since it imposes the condition $\dagger(c) = c$ strictly.

Why higher dagger categories:

RP/unitary TFTs := †-functors $\mathrm{Bord}_n \to \mathrm{Hilb}$

C^*-algebras/categories
higher Hilbert spaces

fully local

†-n-categories

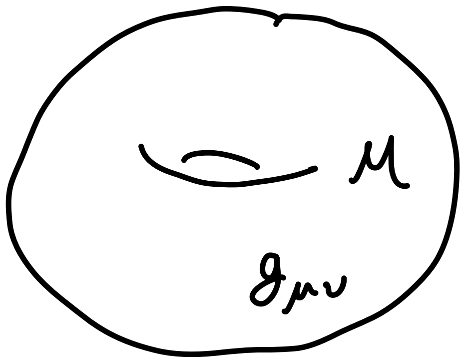
with adjoints
$G \to PL_n/O_n$

without adjoints
$G \to \mathbb{Z}_2^n$

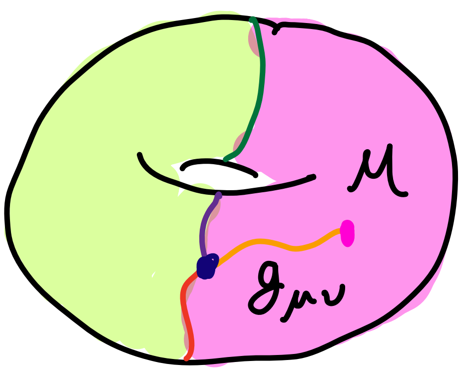
Categories of topological defects

......

Defects in QFT:

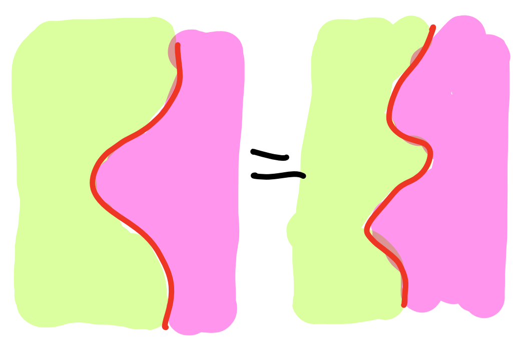


$$\longmapsto Z(M) \in \mathbb{C}$$



$$\longmapsto Z(M, \mathbb{D}) \in \mathbb{C}$$

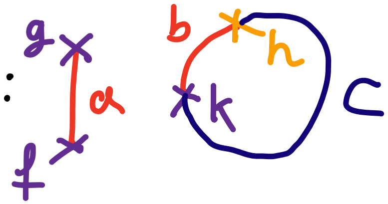
<u>topological</u> defects:



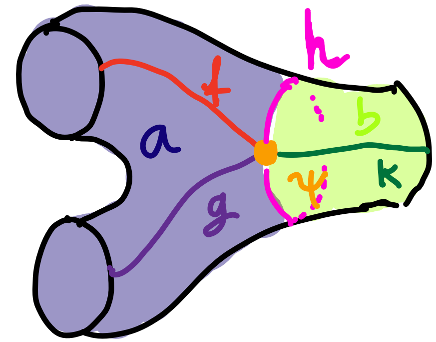
2) Topological defects

$\{D_0, D_1, \dots, D_n, s,t : D_{n-1} \to D_n, \dots\}$

$\mathrm{Bord}_n^{\mathbb{D}}$: Obj: compact stratified $n-1$ dimensional manifolds, labelled by $\mathbb{D}$:



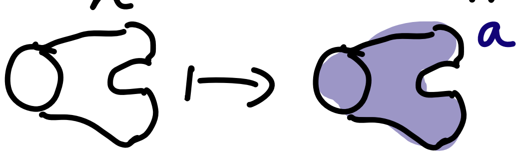
Mor: labelled stratified bordisms



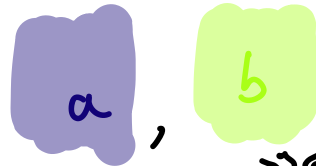
Def. A <u>defect TFT</u> is a symmetric monoidal functor

$Z^{\mathbb{D}} : \mathrm{Bord}_n^{\mathbb{D}} \longrightarrow \mathcal{C}$ ⟵ sym. monoidal target category.

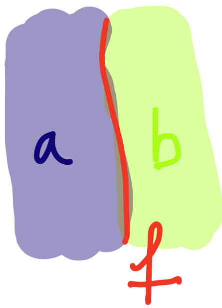
Rem.: $\mathrm{Bord}_n \xhookrightarrow{\iota_a} \mathrm{Bord}_n^{\mathbb{D}}$ $\Rightarrow$ For every label a of top strata there is a corresponding TFT $Z_a := \iota_a^* Z$. (We often identify Z_a and a)

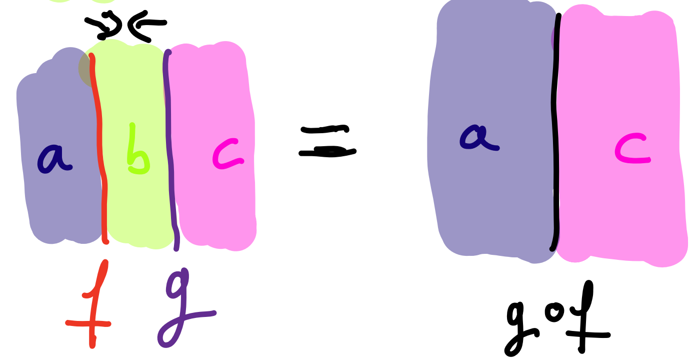


Higher categories of topological defects $\mathcal{D}^{or}_{Z}$ in a QFT:

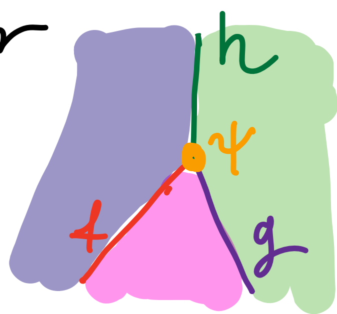
Obj: labels for top strata 

, ...

1-Mor: 

"Composition 

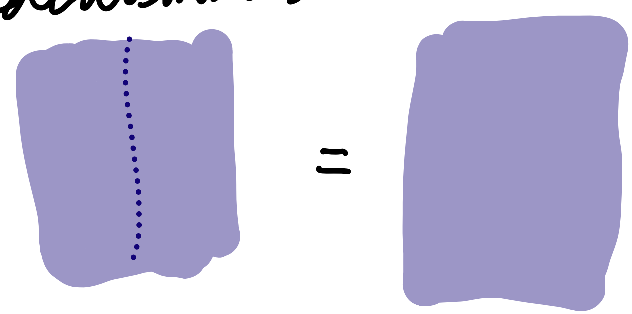
"

2-Mor 

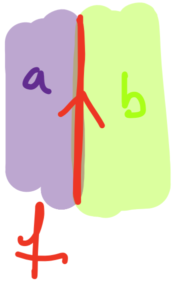
$\psi : g \circ f \Rightarrow h$

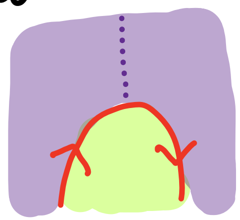
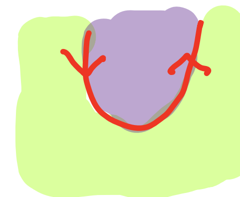
....

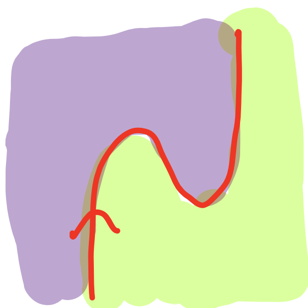
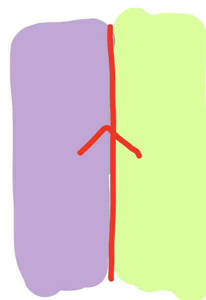
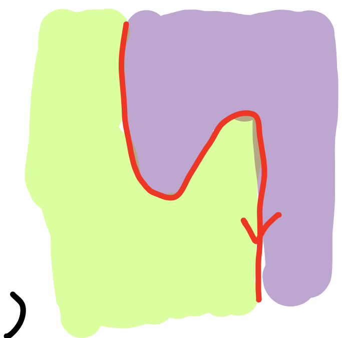
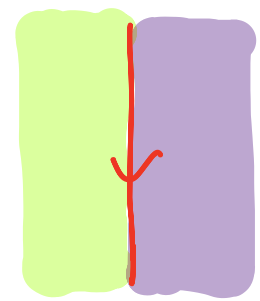
Identities = trivial defects



Extra structure:



$\rightsquigarrow$  $\bar{f} \circ f \Rightarrow id_a$  $id_b \Rightarrow \bar{f} \circ f$

s.t.  =  ,  = 

$\Rightarrow$ All morphisms have adjoints & $f^R = f^L = \bar{f}$

$\Rightarrow \mathcal{D}^{or}_{\mathcal{C}}$ is a "pivotal" n-category

$n=2$ [Davydov, Kong, Runkel '11]

$n=3$ Gray category with adjoints [Barrett, M, S, '12]

[Carqueville, Meusburger, Schaumann '20]

3) Unoriented theories in 1D

Q: What structure do we find for other tangential structures?

1D unoriented TQFTs:

Prop.: sym monoidal functors $\mathrm{Bord}_1^{O(1)} \longrightarrow \mathcal{C}$ are classified by an object $c \in \mathcal{C}$ equipped with a symmetric non-degenerate pairing $\kappa : c \otimes c \longrightarrow \mathbb{1}$.

"Proof": $c = Z(\bullet)$ $\quad \kappa = Z(\supset) : c \otimes c \longrightarrow 1$

symmetric: $Z(\supset) = Z(\propto)$

Rem: This also follows from the CH: $O_1 = \mathbb{Z}_2 \circlearrowright (\mathcal{C}^{f.d.})^\times$ $\quad c \mapsto c^\vee$ $\quad \square$

$$\mathrm{TFT}^{O(1)} \cong (\mathcal{C}^\times)^{\mathbb{Z}_2} \ni \left(c,\ c^\vee \xrightarrow[\sim]{h_c} c,\ \text{s.t.}\ \begin{tikzcd} c \cong c^{\vee\vee} \arrow[r, "(h_c^\vee)^{-1}"] \arrow[rr, bend right, "\mathrm{id}"'] & c^\vee \arrow[r, "h_c"] & c \end{tikzcd} \right)$$

Defects: $\mathbb{D} \leftarrow \{ \; a \; , \; b \; , \; \}$

$$Z : \mathrm{Bord}_1^{\mathbb{D}} \longrightarrow \mathcal{C} \iff (c_a, x_a, c_b, x_b, f : c_a \to c_b)$$

$$Z(\quad) = f : c_a \longrightarrow c_b$$

$$Z(\quad) = Z(\quad) \qquad c_b \xrightarrow{\Delta_a} c_b \otimes c_a \otimes c_a \xrightarrow{id \otimes f \otimes id} c_b \otimes c_b \otimes c_a \xrightarrow{x_b} c_a$$

$$\Rightarrow \mathcal{D}_{\mathcal{C}}^{O(1)} : \quad \mathrm{Obj} : (c, x_c)$$

$$\mathrm{Mor}_{\mathcal{D}}(c, x_c ; c', x_{c'}) = \mathrm{Mor}_{\mathcal{C}}(c, c')$$

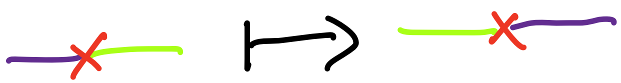
$\mathcal{D}_{\mathcal{C}}^{O(1)} \longrightarrow \mathcal{C}$ which is fully faithful.

$$\begin{tikzcd} \mathcal{D}_{\mathcal{C}}^{O(1)} \arrow[r] \arrow[d] & \mathcal{C} \\ \mathcal{D}_{\mathcal{C}}^{SO(1)} \cong \mathcal{C}^{f.d} \arrow[ur] & \end{tikzcd}$$

For $\mathcal{C} = \mathrm{Vect}^{f.d.}$

$$\begin{tikzcd} \mathrm{Sym\,Vect} \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & \mathrm{Vect} \\ & \mathrm{Vect} \arrow[ur, "\sim"'] & \end{tikzcd}$$

Additional structure:



$$\mathcal{D}_{\mathcal{C}}^{O(1)} \longrightarrow \mathcal{D}_{\mathcal{C}}^{O(1)}$$

$$f : (c,h) \to (c',h') \longmapsto f^\dagger : (c',h') \longrightarrow (c,h)$$

$$c' \xrightarrow{h'^{-1}} c'^\vee \xrightarrow{f^\vee} c^\vee \xrightarrow{h} c$$

Proposition. $\mathcal{D}_{\mathcal{C}}^{O(1)}$ is a $\dagger$-category.

anti-involution

$c \longmapsto c^\vee$

$$\begin{tikzcd}
(\mathcal{C}^\times)^{\mathbb{Z}/2} \arrow[r, "f"] \arrow[d] & \mathcal{C} \arrow[loop right, "\circlearrowleft"] \\
\mathrm{im}(f) \arrow[ur, hook] &
\end{tikzcd}$$

dagger categories $\Leftrightarrow$ [Stehouwer Steinebrunner '23]

Coherent $\dagger$-category:

$\mathcal{C}$ an anti-involutive category

groupoid

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, two heads] \arrow[dr] & \mathcal{C} \\
& (\mathcal{C}^\times)^{\mathbb{Z}/2} \arrow[u]
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
\mathcal{C}_0 \arrow[r, two heads] & \mathcal{C} \\
& \text{anti-involutive} \arrow[u]
\end{tikzcd}$$

4) Higher dagger categories

Unoriented n-dimensional TQFTs: $\left(\begin{array}{l}\mathcal{C} \text{ a sym monoidal} \\ \text{f.d. } n\text{-category.}\end{array}\right)$

CH with defects:

$$ {}_{c_0}\mathcal{C}^{O_n} \longrightarrow {}_{c_1}\mathcal{C}^{O_{n-1}} \longrightarrow {}_{c_2}\mathcal{C}^{O_{n-2}} \cdots \boxed{{}_{c_{n-1}}\mathcal{C}^{O_1} \longrightarrow \mathcal{C}} $$

$\dagger$-category

$\Rightarrow$ O_n-dagger n-category

Other tangential structures:

$$ {}_{c_0}\mathcal{C}^{G_n} \longrightarrow {}_{c_1}\mathcal{C}^{G_{n-1}} \longrightarrow {}_{c_2}\mathcal{C}^{G_{n-2}} \cdots {}_{c_{n-1}}\mathcal{C}^{G_1} \longrightarrow \mathcal{C} $$

$\Rightarrow$ $G_\bullet$-dagger n-category

Thm [Barwick Schommer-Pries '21]
$\mathrm{Aut}(\mathrm{Cat}_{(\infty,n)}) \cong \mathbb{Z}_2^n$, where $(\mathbb{Z}_2)_k$ acts by reversing the direction of k-morphisms.

Def. Let $G \xrightarrow{\rho} \mathbb{Z}_2^n$ be a group homomorphism.
A <u>G-volutive category</u> is a fixed point for the induced G-action on $\mathrm{Cat}_{(\infty,n)}$.

Example: $\mathcal{B} \in \mathrm{Cat}_2$ & $G = \mathbb{Z}_2^2$.
$\psi_1 : \mathcal{B} \xrightarrow{\sim} \mathcal{B}^{1\mathrm{op}}$, $\psi_2 : \mathcal{B} \xrightarrow{\sim} \mathcal{B}^{2\mathrm{op}}$, $\psi_1^{1\mathrm{op}} \circ \psi_1 \cong id$, ...

<u>Note</u>:
$\mathrm{Cat}_{(\infty,0)} \hookrightarrow \mathrm{Cat}_{(\infty,1)} \hookrightarrow \dots \hookrightarrow \mathrm{Cat}_{(\infty,n)}$
Are all $(\mathbb{Z}_2)^n$-equivariant as well as
$$\iota_k : \mathrm{Cat}_{(\infty,n)} \longrightarrow \mathrm{Cat}_{(\infty,k)}$$

Def A <u>flagged fully dagger (∞,n)-category</u> is a chain

$$\mathcal{C}_0 \to \mathcal{C}_1 \to \cdots \to \mathcal{C}_n$$

where:

- $\mathcal{C}_i$ is a $(\mathbb{Z}_2^i)$-volutive category
- the functor $\mathcal{C}_k \to \mathcal{C}_{k+1}$ is $(\mathbb{Z}_2^{k+1})$-volutive and essentially surjective on $(\leq k)$-morphisms [Ayala Francis '18]

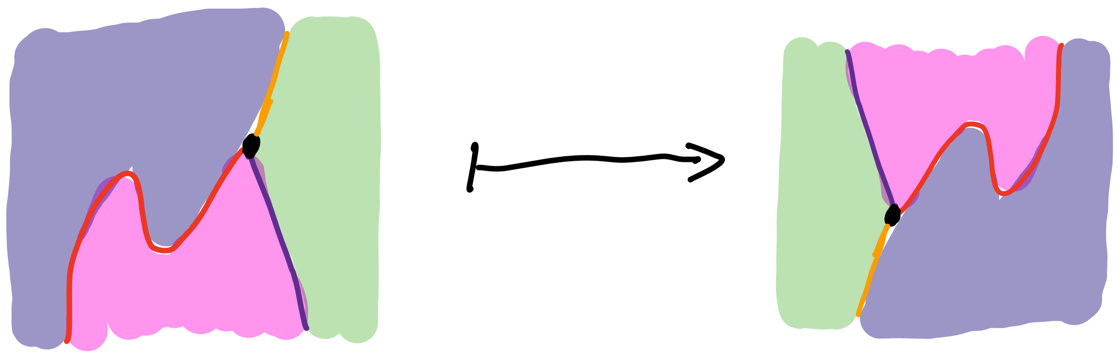
$$\Longleftrightarrow \quad \begin{tikzcd} \mathcal{C}_k \arrow[r] \arrow[dr, dotted] & \mathcal{C}_{k+1} \\ & \left(\iota_k \mathcal{C}_{k+1}\right)^{(\mathbb{Z}_2)_k} \arrow[u] \end{tikzcd} \quad \text{factoring through } \left(\iota_k \mathcal{C}_k\right)^{(\mathbb{Z}_2)_{k+1}}$$

Def. A <u>fully dagger (∞,n)-category</u> is a flagged fully dagger category such that $\mathcal{C}_k \to \left(\iota_k \mathcal{C}_{k+1}\right)^{\mathbb{Z}_2}$ is fully-faithful on $(>k)$-morphisms.

†-categories with adjoints

Conj: $\mathrm{Aut}(\mathrm{AdjCat}_{(\infty,n)}) \simeq PL_n \supseteq O_n$

framed graphical calculus:



For $n=2$:

$PL_2 \simeq O_2 = S^1 \rtimes \mathbb{Z}_2 \simeq B\mathbb{Z} \rtimes \mathbb{Z}_2$

$$\mathcal{B} \mapsto \mathcal{B}^{2op} \underset{\tau}{\simeq} \mathcal{B}^{1op}$$

$$f \mapsto f^R$$

$$\begin{tikzcd} \mathrm{AdjCat}_2 \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow (-)^{RR}"] & \mathrm{AdjCat}_2 \end{tikzcd}$$

$$(-)^{RR}|_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}$$

$$f \longmapsto f^{RR}$$

Def. Let $G \xrightarrow{\rho} O_n$ be a group homomorphism.
A <u>G-volutive category</u> is a fixed point for the induced G-action on $\mathrm{AdjCat}_{(\infty,n)}$.

Note:
$\mathrm{Adj}\,\mathrm{Cat}_{(\infty,n)} \xrightarrow{c_k} \mathrm{Adj}\,\mathrm{Cat}_{(\infty,k)}$ for $k \leq n$ is $O(k) \times O(n-k)$-equivariant.

Def.: An <u>O_n-dagger (∞,n)-category</u> is a chain

$$\mathcal{C}_0 \to \mathcal{C}_1 \to \dots \to \mathcal{C}_n$$

when $\mathcal{C}_i$ is an O_i-volutive category & $\mathcal{C}_i \to \mathcal{C}_{i+k}$ is $O_i \times O_k$-volutive and essentially surjective on $(\leq i)$-morphisms & $\mathcal{C}_i \to (c_i \mathcal{C}_{i+1})^{O_1}$ is fully faithful on $(> k)$-mor.

Consider a sequence of groups $\mathcal{G}_\bullet$ with compatible maps to

$$O_\bullet : \quad \begin{tikzcd} \mathcal{G}_1 \arrow[r] \arrow[d, "\rho_1"] & \mathcal{G}_2 \arrow[r] \arrow[d, "\rho_2"] & \cdots \arrow[r] & \mathcal{G}_n \arrow[d, "\rho_n"] \\ O_1 \arrow[r] & O_2 \arrow[r] & \cdots \arrow[r] & O_n \end{tikzcd}$$

$\mathcal{G}_\bullet$- dagger category:

$\mathcal{C}_0 \to \mathcal{C}_1 \to \cdots \to \mathcal{C}_n$, $\quad \mathcal{C}_i$ is $\mathcal{G}_i$-volutive.

$\mathcal{C}_i \longrightarrow \mathcal{C}_{i+k}$ is $\mathcal{G}_i \times \rho_{i+k}^{-1}(O_k)$-volutive.

Examples

$$\begin{tikzcd} SO_1 \arrow[r] \arrow[d] & SO_2 \arrow[r, dash, dashed] \arrow[d] & SO_n \arrow[d] \\ O_1 \arrow[r] & O_2 \arrow[r, dash, dashed] & O_n \end{tikzcd} \qquad \begin{tikzcd} Spin_1 \arrow[r] \arrow[d] & Spin_2 \arrow[r] \arrow[d] & \cdots & Spin_n \arrow[d] \\ O_1 \arrow[r] & O_2 \arrow[r] & \cdots & O_n \end{tikzcd}$$

5) Back to defects

Proposal: topological defects of type $(\mathcal{G}_\bullet \to O_\bullet)$ in a QFT form an $\mathcal{G}_\bullet$-dagger category

true in 1D

"true" in 2D (work in progress)

Rest of the talk:

SO_2-dagger-bicat $\iff$ pivotal bicategories

Sketch of a justification

SO_2-dagger bicategories

SO_2-volutive: $\mathcal{B}$ a bicategory with adjoints + natural isomorphism $S : id_{\mathcal{B}} \Rightarrow (-)^{RR}$, s.t. ...

SO_2-dagger:

$$\begin{tikzcd}
 & \mathcal{B}_0 \arrow[r, "f"] \arrow[dr] & \mathcal{B} \\
\text{2-groupoids} \arrow[ur] \arrow[rr] & & c_0\mathcal{B}^{SO_2} \arrow[u]
\end{tikzcd}$$

An object of $\mathcal{B}_0$ is completly determind by its image in $(c_0\mathcal{B})^{SO_2}$: $(b, \psi_b : S_b \overset{\sim}{\Rightarrow} id_b)$

$\rightsquigarrow \mathcal{B}'$: $Obj(\mathcal{B}') = Obj(\mathcal{B}_0)$

$Mor_{\mathcal{B}'}(b_0, b_0') = Mor_{\mathcal{B}}(f(b_0), f(b_0'))$

$S' : id_{\mathcal{B}'} \Rightarrow (-)^{RR}$: $S'_{(b,\psi_b)} = id_b$,

$$\begin{tikzcd}
(b,\psi_b) \arrow[r, "S_b"] \arrow[r, bend left=50, "id_b"] \arrow[r, phantom, bend left=25, "\Downarrow \psi_b^{-1}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow S_f"] & (b,\psi_b) \arrow[d, "f^{RR}"] \\
(b',\psi_{b'}) \arrow[r, "S_{b'}"] \arrow[r, bend right=50, "id_{b'}"'] \arrow[r, phantom, bend right=25, "\Downarrow \psi_{b'}"] & (b',\psi_{b'})
\end{tikzcd}$$

$(\mathcal{B}', S')$ is a pivotal bicategory

Some justifications for TFTs

Conj: The $O_n \curvearrowright \mathrm{SymRigidCat}_{(\infty,n)}$ has a canonical trivialization.

$\Rightarrow$ Every rigid sym monoidal (∞,n)-category $\mathcal{C}$ has a canonical O_n-volution.

Conj: The induced O_{n-k} action on $\iota_k \mathcal{C}$ agrees with the action in the CH with defects.

CH defects $\Rightarrow$ Proposal

Outlook:

- Connection to RP/unitarity
- Condensation / Orbifold completion
- Description of O_2, $Spin_2$, ... dagger bicategories

unitary duality

$f^{RRRR} = f$

[Spin statistics $f^{RR} = (-1)^F$]

- Many open questions!

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Thank you for your attention!