An Introduction to Quantum Representations of Mapping Class Groups

Lukas Woike
Université de Bourgogne

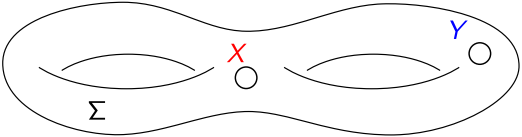
Based on different joint projects with Adrien Brochier (IMJ-PRG), Lukas Müller (Perimeter Institute), Christoph Schweigert (Hamburg) and Yang Yang (Leipzig)

TQFT Club Seminar in Lisbon (online)

January 24, 2024

Modular functors

Following [Segal 88, Moore-Seiberg 88, Turaev 94, Tillmann 98, Bakalov-Kirillov 01, ...].



$$\mathrm{Map}(\Sigma) = \pi_0(\mathrm{Diff}(\Sigma)) \; ; \quad \text{Example: } \mathrm{Map}(\mathbb{S}^1 \times \mathbb{S}^1) \cong \mathrm{SL}(2, \mathbb{Z}) \; .$$

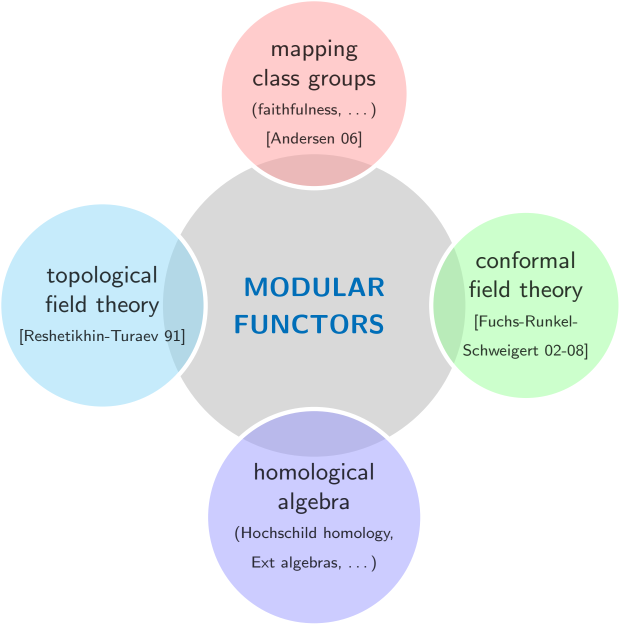
$$(\Sigma; X, Y, \ldots) \longmapsto \text{vector space } B(\Sigma; X, Y, \ldots) \curvearrowleft \mathrm{Map}(\Sigma)$$

for all surfaces, compatible with the gluing of surfaces. The vector space $B(\Sigma; X, Y, \ldots)$ is called *space of conformal blocks*.

Formal definition using modular operads in the sense of Getzler-Kapranov

A *modular functor* is a modular algebra over the modular surface operad (or a certain central extension of it) with values in a symmetric monoidal bicategory of linear categories.

picture: arXiv:2201.07542


mapping
class groups
(faithfulness, . . .)
[Andersen 06]
topological
field theory
[Reshetikhin-Turaev 91]
MODULAR
FUNCTORS
conformal
field theory
[Fuchs-Runkel-
Schweigert 02-08]
homological
algebra
(Hochschild homology,
Ext algebras, . . .)

The classical construction three-dimensional topological field theories

Theorem [Bartlett-Douglas-Schommer-Pries-Vicary 15]

Once-extended three-dimensional topological field theories are equivalent to semisimple modular categories. (The topological field theory associated to a semisimple modular category is the Reshetikhin-Turaev construction.)

Once-extended $=$ defined up to codimension two
$\Longrightarrow$ Restriction to surfaces gives us a modular functor.

- A *finite tensor category* [Etingof-Ostrik] $\mathcal{A}$ over some algebraically closed field k is
  - a linear abelian category $\mathcal{A}$ with finite-dimensional morphism spaces, enough projective objects, finitely many isomorphism classes of simple objects such that every object has finite length,
  - with a monoidal product $\otimes : \mathcal{A} \boxtimes \mathcal{A} \longrightarrow \mathcal{A}$,
  - a rigid duality $-^{\vee}$,
  - and simple unit.
- A *braiding* on a monoidal category is a natural isomorphism $X \otimes Y \longrightarrow Y \otimes X$ subject to the hexagon axioms. A braiding on a finite tensor category is called *non-degenerate* if the only objects that trivially double braid with all other objects are finite direct sums of the monoidal unit.

- A *balancing* on a braided monoidal category is a natural isomorphism $\theta_X : X \longrightarrow X$ subject to

$$\theta_{X\otimes Y} = c_{Y,X} c_{X,Y} (\theta_X \otimes \theta_Y) \ ,$$
$$\theta_I = \mathrm{id}_I \ .$$

- If in presence of duality we have additionally

$$\theta_{X^\vee} = \theta_X^\vee \ ,$$

  we call the balancing a *ribbon structure*.

- *Modular category*: finite ribbon category with non-degenerate braiding.

Sources for modular categories

Certain Hopf algebras ($\longrightarrow$ quantum groups) and vertex operator algebras ($\longrightarrow$ two-dimensional conformal field theory).

If $\mathcal{A}$ is a semisimple modular category, the *space of conformal blocks* for the surface with genus g and n boundary components is

$$\mathcal{A}(I, X_1 \otimes \cdots \otimes X_n \otimes \mathbb{F}^{\otimes g}) \quad \text{for} \quad X_1, \ldots, X_n \in \mathcal{A}$$

with $\mathbb{F} = \bigoplus_{\text{basis of simples}} X_i^\vee \otimes X_i$.

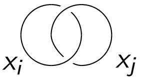
An illustration for the torus

For a complex semisimple modular category $\mathcal{A}$, the space of conformal blocks of the torus is spanned by the isomorphism classes $[x_0], \ldots, [x_n]$ of simple objects. Consider the generators

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} , \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for the mapping class group $\mathrm{SL}(2, \mathbb{Z})$ of the torus. Then:

- T acts diagonally, namely by $\theta_{x_i} \in k$ on $[x_i]$.
- S acts by the so-called 'S-matrix' with entries:



.

Theorem [Schauenburg-Ng 2010]

The kernel of this $SL(2, \mathbb{Z})$-representation is a congruence subgroup whose level is the order of the ribbon twist θ.

The non-semisimple improvement

The construction still works with the *coend* $\mathbb{F} = \int^{X \in \mathcal{A}} X^\vee \otimes X$ instead — even beyond semisimplicity!

Theorem [Lyubashenko 95]

If $\mathcal{A}$ is a (possibly non-semisimple!) modular category, the vector spaces $\mathcal{A}(I, X_1 \otimes \cdots \otimes X_n \otimes \mathbb{F}^{\otimes g})$ carry projective mapping class group actions.

Problem: How can we approach the search for *all* mapping class group systematically and based on a solid topological underpinning?

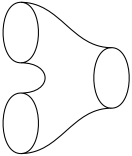
Goal: Classification of modular functors

Preliminary observation: The boundary labels form the objects of a linear category, the *circle category*, that we denote by $\mathcal{A}$.

(I will present the situation in which $\mathcal{A}$ is finitely cocomplete and $B(\Sigma, -)$ cocontinuous in the labels. Technically speaking: We work in $\mathsf{Rex}^{\mathsf{f}}$.)

Genus zero modular functors

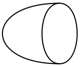
[Wahl 01, Salvatore-Wahl 03]



$\longmapsto \quad \otimes : \mathcal{A} \boxtimes \mathcal{A} \longrightarrow \mathcal{A}$

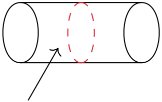
monoidal product

plus braiding $c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$



$\longmapsto \quad I \in \mathcal{A}$

monoidal unit



$\longmapsto \quad \theta : \mathrm{id}_{\mathcal{A}} \Longrightarrow \mathrm{id}_{\mathcal{A}}$

balancing

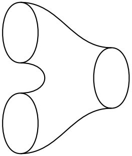
Dehn twist

$$\theta_{X \otimes Y} = c_{Y,X} c_{X,Y} (\theta_X \otimes \theta_Y)$$

$$\theta_I = \mathrm{id}_I$$

Genus zero modular functors

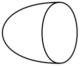
[Wahl 01, Salvatore-Wahl 03]



$\longmapsto \quad \otimes : \mathcal{A} \boxtimes \mathcal{A} \longrightarrow \mathcal{A}$

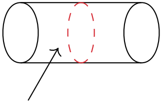
monoidal product

plus braiding $c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$



$\longmapsto \quad I \in \mathcal{A}$

monoidal unit



Dehn twist

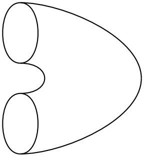
$\longmapsto \quad \theta : \mathrm{id}_{\mathcal{A}} \Longrightarrow \mathrm{id}_{\mathcal{A}}$

balancing

$$\theta_{X \otimes Y} = c_{Y,X} c_{X,Y} (\theta_X \otimes \theta_Y)$$
$$\theta_I = \mathrm{id}_I$$

[Müller-W. 20-22]

cyclic structure



ribbon Grothendieck-Verdier duality in the sense of Boyarchenko-Drinfeld

$$D : \mathcal{A} \xrightarrow{\cong} \mathcal{A}^{\mathrm{opp}}$$

$$\mathrm{Hom}_{\mathcal{A}}(- \otimes Y, K) \cong \mathrm{Hom}_{\mathcal{A}}(-, DY)$$

with $K := DI$

$$\theta_{DX} = D\theta_X$$

'genus zero modular functors = ribbon Grothendieck-Verdier categories'

Extension to higher genus

Theorem [Brochier-W. 22]

Any genus zero modular functor (including the cyclic structure) extends to higher genus in at most one way, up to a contractible choice.

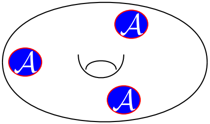
For the extension, there will be an obstruction — and this is exactly what we can express in terms of **factorization homology***!*

What is *factorization homology*? [Beilinson-Drinfeld, Lurie, Ayala-Francis, . . . ; 2000-]

coefficients: E_2-algebra, e.g. braided category

$$\int_{\Sigma} \mathcal{A} = \bigoplus_{\substack{\sqcup_n \mathbb{D}^2 \hookrightarrow \Sigma \\ n \geq 0}} \mathcal{A}^{\boxtimes n} \Big/ \sim$$

surface



Skein module functors for handlebodies

Take a surface Σ with n boundary components and choose a handlebody filling H. If $\mathcal{A}$ is a ribbon Grothendieck-Verdier category, then $\mathcal{A}$ extends uniquely to all handlebodies ('*ansular functor*' [Müller-W.]).

One may show that it produces a functor

$$\Phi_{\mathcal{A}}(H) : \int_{\Sigma} \mathcal{A} \longrightarrow \mathcal{A}^{\boxtimes n} \ .$$

Definition

We say that $\mathcal{A}$ is *connected* if $\Phi_{\mathcal{A}}(H) \cong \Phi_{\mathcal{A}}(H')$ for all handlebodies H and H' with boundary Σ, and all surfaces Σ (isomorphism of module functors).

This condition can be reduced to genus one.

More details on the construction of generalized skein modules [optional]

Theorem [Müller-W. 2022]

Genus zero restriction provides an equivalence between ansular functors and cyclic framed E_2-algebras.
In $\mathrm{Rex}^{\mathrm{f}}$*, the ansular functor associated to a ribbon Grothendieck-Verdier category $\mathcal{A}$ sends a handlebody of genus g and n boundary components labeled with $X_1, \ldots, X_n$ to the hom space*

$$\mathcal{A}(X_1 \otimes \cdots \otimes X_n \otimes \mathbb{A}^{\otimes g}, K)^*$$

defined using the canonical end $\mathbb{A} = \int^{X \in \mathcal{A}} X \otimes DX$ (D is the duality functor of $\mathcal{A}$).

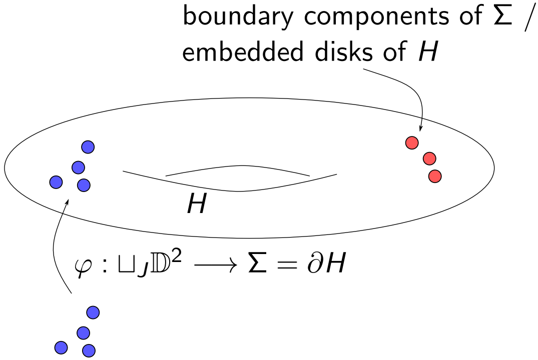
Uses a result of Giansiracusa on the *derived modular envelope* of framed E_2.

More details on the construction of generalized skein modules [optional]

Let $\mathcal{A}$ be a cyclic framed E_2-algebra in $\mathrm{Rex}^{\mathrm{f}}$.

- For a handlebody H with $\partial H = \Sigma$ (the n embedded disks of H are converted in boundary components of Σ), consider an embedding $\varphi : \sqcup_J \mathbb{D}^2 \longrightarrow \Sigma$. This endows H with $m := |J|$ more embedded disks in its boundary. We denote this handlebody by H^φ.
- By evaluation of the ansular functor $\widehat{\mathcal{A}}$ associated to $\mathcal{A}$, we get a 1-morphism

$$\mathcal{A}^{\boxtimes m} \xrightarrow{\widehat{\mathcal{A}}(H^\varphi)} \mathcal{A}^{\boxtimes n}$$



- This is natural in φ and hence produces the desired 1-morphism

$$\Phi_{\mathcal{A}}(H) : \int_{\Sigma} \mathcal{A} = \operatorname*{hocolim}_{\varphi : \sqcup_J \mathbb{D}^2 \longrightarrow \Sigma} \mathcal{A}^{\boxtimes J} \longrightarrow \mathcal{A}^{\boxtimes n} \ .$$

picture: arXiv:2212.11259

Classification of modular functors

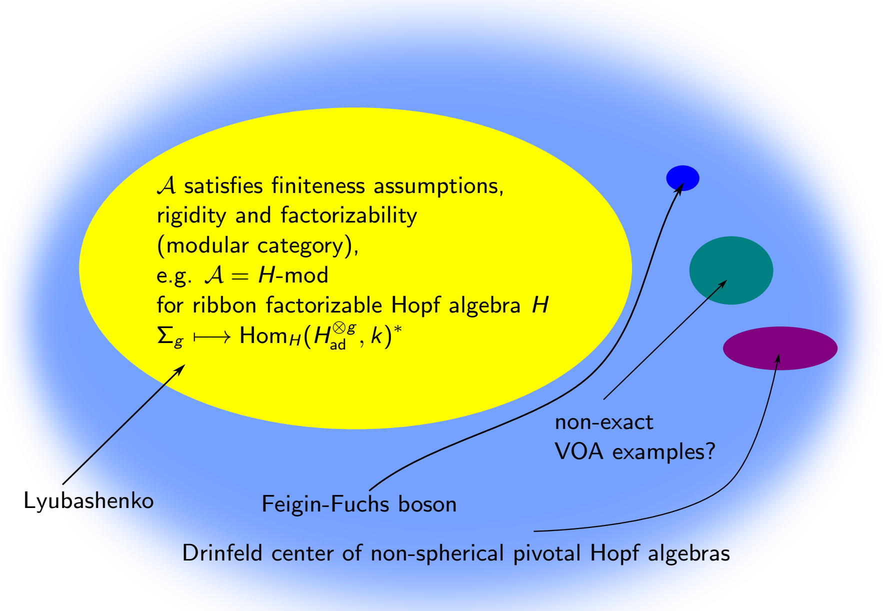
One can define a *moduli space $\mathfrak{MF}$ of modular functors*.

Theorem [Brochier-W. 22]

Genus zero restriction provides a homotopy equivalence from the moduli space $\mathfrak{MF}$ of modular functors to connected ribbon Grothendieck-Verdier categories.

The closed surface of genus g is always sent to $\mathrm{Hom}_{\mathcal{A}}(\mathbb{A}^{\otimes g}, K)^*$ with the *canonical end* $\mathbb{A} = \int_{X \in \mathcal{A}} X \otimes DX$ and $K = DI$.

Moduli space of modular functors



Application / special case I: Drinfeld centers

For a pivotal finite tensor category $\mathcal{C}$, denote by α the *distinguished invertible object* that describes the quadruple dual via the *Radford isomorphism* of Etingof-Nikshych-Ostrik

$$-^{\vee\vee\vee\vee} \cong \alpha \otimes - \otimes \alpha^{-1} \ .$$

Theorem [Müller-W. 2022]

The distinguished invertible object $\alpha \in \mathcal{C}$, equipped with a suitable half braiding, is a dualizing object in the Drinfeld center $Z(\mathcal{C})$ that makes $Z(\mathcal{C})$ a ribbon Grothendieck-Verdier category.

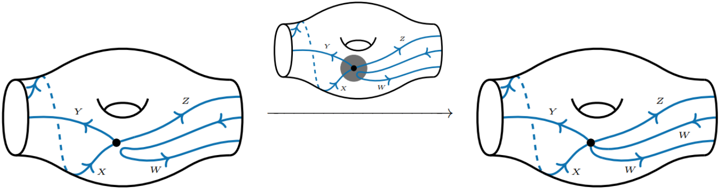
This ribbon Grothendieck-Verdier category is connected; it therefore gives rise to a modular functor, even when $Z(\mathcal{C})$ is not a modular category in the traditional sense.
(This will happen when $\mathcal{C}$ is not spherical in the sense of Douglas-Schommer-Pries-Snyder by a result of [Shimizu 17].)

Theorem [Müller-Schweigert-W.-Yang 2023]

For a pivotal finite tensor category $\mathcal{C}$, the modular functor for $Z(\mathcal{C})$ admits a string-net description:

$$\mathsf{SN}_{\mathcal{C}} \simeq \mathfrak{F}_{Z(\mathcal{C})} \ .$$

($\mathsf{SN}_{\mathcal{C}}$ is constructed by applying the Levin-Wen type string-net construction to the tensor ideal $\mathrm{Proj}\,\mathcal{C}$ and finite free cocompletion.)



This provides a new description for $\mathfrak{F}_{Z(\mathcal{C})}$ and shows that $\mathfrak{F}_{Z(\mathcal{C})}$ extends to an open-closed modular functor and is anomaly-free.
picture: arXiv:2312.14010

Applications II: Dehn twists

[Blanchet-Costantino-Geer-Patureau-Mirand 16] & [De Renzi, Gainutdinov, Geer, Patureau-Mirand, Runkel 2020] prove that the modular functor for a certain ribbon factorizable Hopf algebra (the 'small quantum group') has the following property: For a closed surface, any Dehn twist acts by an automorphism of infinite order. We generalize this as follows:

Theorem [Müller-W. 2023]

Let H be a ribbon factorizable Hopf algebra whose ribbon element has order $N \in \mathbb{N} \cup \{\infty\}$. On the space of conformal blocks that the modular functor for H-mod assigns to Σ_g with $g \geq 1$, any Dehn twist about an essential simple closed curve acts by an automorphism of order N if the curve is non-separating.

If the curve is separating, this is generally false, but we can also give an expression for the order.

Applications II: Dehn twists

We use this to prove:

- The modular functor for H annihilates all Johnson kernels (subgroups generated all Dehn twists about separating simple closed curves) if and only if the ribbon element acts trivially on H_{ad} and if H_{ad} trivially double braids with itself.
- The modular functor for H annihilates all Torelli groups (subgroups acting trivially on first homology) if and only if H is abelian; this is a generalization of a result of [Fjelstad-Fuchs 20].

Applications III: Vertex operator algebras

[Allen-Lentner-Schweigert-Wood 21] provide some very mild conditions on a vertex operator algebra V and a notion of module making V-modules a ribbon Grothendieck-Verdier category (possibly with a non-exact monoidal product!).

Theorem [Müller-W. + Brochier-W. 22]

Under the above assumptions, the genus zero spaces of conformal blocks

$$\Sigma_{0,n} \longmapsto \mathrm{Hom}_V(- \otimes \cdots \otimes -, V^*)^*$$

on which the ribbon braid groups act through the ribbon Grothendieck-Verdier structure have a unique extension to all handlebodies (an 'ansular functor') and at most one extension to a modular functor (both with values in Rex*).*

This will hopefully pave the way to comparison theorems to constructions of conformal blocks directly from the vertex operator algebra, see [Ben-Zvi-Frenkel] and more recently [Damiolini-Gibney-Tarasca].