Super period map and projectedness of supermoduli

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- 1980's: Super Riemann surfaces (SRS) are introduced by string theorist as the worldsheets of propagating superstrings. Many important quantities in superstring theory are defined as integrals over the moduli spaces $\mathfrak{M}_g$ of SRSs, called *supermoduli space*, e.g., the g-superstring scattering amplitude,

$$\int_{\mathfrak{M}_g} \mu_g,$$

the μ_g is called the *supermeasure*.
- 2002: D'Hoker + Phong compute $g = 2$ scattering amplitude, writing down μ_2 using an analytic formula for the $g = 2$ super period matrix. Basic idea: Write down super period matrix, observe it defines a projection $\mathfrak{M}_2^+ \to \mathcal{SM}_2^+$, compose with the forgetting map $\mathcal{SM}_2^+ \to \mathcal{M}_2$, to get the map

$$\mathfrak{M}_2^+ \to \mathcal{SM}_2^+ \to \mathcal{M}_2 (= \mathcal{A}_2)$$

and then integrate the usual string-measure over $\mathcal{M}_2$ followed by an integration over the fibers.

- 2014: Donagi and Witten prove that $\mathfrak{M}_g$ is not projected for $g \geq 5$. Thus the key fact in D'Hoker and Phong's calculation for $g = 2$ amplitude, namely that $\mathfrak{M}_2$ is projected does not generalize.
- 2013: Witten generalizes D'Hoker and Phong's analytic formula for the super period matrix to higher genus. His formulas show possible poles in the super period matrix, implying poles in μ_g, along a certain locus $\mathcal{B}$ in $\mathfrak{M}_g$, now called the bad locus (divisor). He conjectures that poles only appear after genus twelve.
- 2020: Proof that super period matrix is regular for $g \leq 11$, verifying Witten's conjecture.
- Today: Supermoduli space *is* projected away from $\mathcal{B}$ for all genus g. Furthermore, at least for $g = 2$ and $g = 3$ the super period matrix defines a projection from the complement, U_g, of $\mathcal{B}$ onto is bosonic reduction $U_{g,bos} \subset \mathcal{SM}_g$, and using the regularity result above, the projection extends to a projection $\mathfrak{M}_3^+ \to \mathcal{SM}_3^+$, and similarly for $g = 2$. In particualr, we find that $\mathfrak{M}_3^+$ is projected. This is joint work with Ron Donagi.

- *Super vector space*: A vector space V with a $\mathbb{Z}_2$-grading,

$$V = V_0 \oplus V_1.$$

Example: $\mathbb{C}^{1|1}$: Take $\mathbb{C}^2$ and a basis e_1, e_2. Then $e_1, \Pi e_2$ is a basis for $\mathbb{C}^{1|1}$. Here Π is the parity-reversing functor: If $e_1, \ldots, e_n$ generate $\mathbb{C}^n$, then $\Pi e_1, \ldots, \Pi e_n$ generate $\mathbb{C}^{0|n}$

- *Superalgebra*: A super vector space V with a multiplication, $V \otimes V \to V$, such that $v \otimes w = (-1)^{|v||w|} w \otimes v$.

Grassmann algebra: Let V be an ordinary k-vector space, and let $e_1, \ldots, e_n$ be a basis, and set $\theta_i := \Pi e_i$. Then,

$$G(V) := \wedge^{\bullet} \Pi V \cong k[\theta_1, \ldots, \theta_n]$$

where $\theta_i \theta_j := \theta_i \wedge \theta_j$, and

$$G(V)_0 = \wedge^{2n} \Pi V, \quad G(V)_1 = \wedge^{2n+1} \Pi V$$

- A *superringed space* is a pair $X = (|X|, \mathcal{O}_X)$, a sheaf of superalgebras $\mathcal{O}_X = \mathcal{O}_{X,0} \oplus \mathcal{O}_{X,1}$ on a topological space $|X|$ such that $\mathcal{O}_X(U)$ is a superalgebra for all open subsets $U \subset |X|$.
- $\mathbb{C}^{m|n}$ is the top space $|\mathbb{C}^m|$ with the sheaf

$$\mathcal{O} : U \subset \mathbb{C}^m \mapsto \mathrm{hol}(U) \otimes G_n(U).$$

  Here G_n is the Grassmann algebra generated by a global frame for the trivial bundle of rank n on $\mathbb{C}^m$. The coordinates on $\mathbb{C}^{m|n}$ are for example $(z_1, \ldots, z_m, \theta_1, \ldots, \theta_n)$.
- *Complex supermanifold*: A superring space $X = (|X|, \mathcal{O}_X)$ locally isomorphic to the local model, $\mathbb{C}^{m|n}$.

Reductions

Let $J := \langle \mathcal{O}_{X,1} \rangle \subset \mathcal{O}_X$. Then

- Bosonic truncation: $X_{ev} = (|X|, \mathcal{O}_{X,0})$, and $\mathcal{O}_{X,0} \subset \mathcal{O}_X$ induces a projection $X \to X_{ev}$
- Bosonic reduction: $X_{bos} = (|X|, \mathcal{O}_X/J)$, and the morphism $\mathcal{O}_X \to \mathcal{O}_X/J$ induces an embedding $X_{bos} \subset X$.
- X is said to be *projected* if the embedding has a section, i.e., a projection $X \to X_{bos}$, and *split* if the projection has linear fibers. Specifically, X is split if it is isomorphic to the normal bundle $N_{X_{bos}/X}$ to $X_{bos} \subset X$.

REMARK: The bosonic reduction is an ordinary manifold, and the bosonic truncation is an ordinary scheme. For example, the bosonic reduction of $\mathbb{C}^{1|2}$ is $\mathbb{C}^1$, the bosonic truncation is isomorphic to the non-reduced scheme $\operatorname{Spec} \mathbb{C}[z, \theta_1\theta_2]$.
REMARK: The bosonic reduction and truncation are the same for $m|1$-dimensional supermanifolds.

Supercurve

A supercurve X is a compact, connected $1|1$-dimensional complex supermanifold. Properties:

- X is locally isomorphic to $\mathbb{C}^{1|1} \cong \operatorname{Spec} \mathbb{C}[z, \theta]$.
- The bosonic reduction (=bosonic truncation) of X is an ordinary curve (Riemann surface), $C := X_{bos}$.

Examples:

- $\mathbb{C}^{1|1}$ (not compact)
- $\mathbb{P}^{1|1}$ (genus $g = 0$ supercurve).
- $\mathbb{WP}^{1|1}(1, 1|m)$, weighted superprojective spaces, (genus $g = 0$ supercurve)
- A branched cover of a genus g_0 supercurve is a supercurve with genus g determined by the usual Hurwitz formula, a function of g_0, and the number of branch and ramification points.

Calculus

A derivation, or vector field, on a supercurve X is a $\mathbb{C}$-linear morphism $\delta : \mathcal{O}_X \to \mathcal{O}_X$ satisfying the super Leibniz rule,

$$\delta(fg) = \delta(f)g + (-1)^{|\delta||f|} f\delta(g).$$

The tangent bundle T_X on X is of rank $1|1$ and thus locally generated by one even and one odd derivation. For example, in local coordinates (z, θ),

1. $\partial_z, \partial_\theta$
2. $\partial_z, \partial_\theta + \theta\partial_z$

To check the parity is as claimed:

$$\partial_z(z) = 1,\ \partial_z(\theta) = 0,$$

$$\partial_\theta(z) = 0,\ \partial_\theta(\theta) = 1$$

Calculus

The tangent bundle T_X comes has a Lie bracket,

$$[\ ,\ ] : T_X \otimes T_X \to T_X,$$

satisfying $[V, W] = VW - (-1)^{|V||W|} WV$. If $[V, W] \neq 0$, then $|V| = |W| = 1$. For example,

$$[\partial_\theta + \theta\partial_z, \partial_\theta + \theta\partial_z] = \partial_z.$$

Compare this to the Lie bracket on the tangent bundle on an ordinary curve, which is always equal to zero.

This gives rise to a special structure on a supercurve, not seen in the ordinary case, called a *superconformal structure*.

Super Riemann surfaces

A superconformal structure $\mathcal{D}$ on a supercurve X is a rank $0|1$ sub-bundle $\mathcal{D}$ of T_X which is maximally non-integrable in the sense that the supercommutator defines an isomorphism

$$[\ ,\ ] : \mathcal{D} \otimes \mathcal{D} \cong T_X/\mathcal{D}.$$

In local coordinates, (z, θ), $\mathcal{D}$ is generated by the odd vector field, $\partial_\theta + \theta\partial_z$, and

$$[\partial_\theta + \theta\partial_z, \partial_\theta + \theta\partial_z] = \partial_z$$

from which we get the saying that $\mathcal{D}$ is a square-root of the derivative.

A *super Riemann surface* is a pair $(X, \mathcal{D})$ of a supercurve X and a superconformal structure $\mathcal{D}$ on X.

Super Riemann surface

We can also take the bosonic reduction of the superconformal structure $\mathcal{D}$ by taking quotient with $J = \langle \mathcal{O}_{X,1} \rangle$, the result,

$$\mathcal{D}_{bos} := \mathcal{D}/J\mathcal{D}.$$

is a $\mathcal{O}_C = \mathcal{O}_X/J$-module, or a rank $0|1$-vector bundle on C. In particular, the parity-reversal $L_\Pi := \Pi\mathcal{D}_{bos}$ is a line bundle on C.

It is a standard exercise to show that $L := L_\Pi^\vee$ is a *spin structure* on C, i.e., L is a line bundle with an identification

$$L^2 = \Omega^1_C.$$

Furthermore, the super Riemann surface X can be recovered from the *spin curve*, (C, L), as the total space of ΠL. This gives rise to the well-known one-to-one correspondence

$$\{ \text{ spin curves } \} = \{ \text{ super Riemann surfaces } \}$$

BUT... this correspondence does not hold for *families of super Riemann surfaces*.

A family of super Riemann surface is a morphism $\pi : X \to S$ of supercurves, with a superconformal structure $\mathcal{D}$ now a rank $0|1$ subbundle of the relative tangent bundle $T_{X/S}$.

FACT: If S is purely bosonic, then $X \to S$ is the total space of a spin structure on a family of spin curves, and we recover the one-to-one correspondence in the previous slide, i.e., for an ordinary scheme S,

$$\{ \text{ fams of SRSs over } \ S \ \} = \{ \text{ fams of spin curves over } S \ \}.$$

FACT: If S has odd coordinates, then $X \to S$ is, in general, not a family of spin curves. In general,

$$\{\text{fams of spin curves}\} \subset \{\text{fams of SRS}\}$$

The geometric object parameterizing families of genus g super Riemann surfaces is called supermoduli space, $\mathfrak{M}_g$. We assume that $g \geq 2$, so that we can treat $\mathfrak{M}_g$ as a supermanifold (it is really a smooth Deligne-Mumford superstack, representing the moduli functor

$$\mathfrak{M}_g : \text{sSch} \to \text{groupoid},$$

. we treat $g = 0, 1$ separately because it is only an Artin superstack).

The one-to-one correspondence between spin curves and super Riemann surfaces over ordinary schemes identifies $(\mathfrak{M}_g)_{bos} : \text{Sch} \to \text{groupoid}$ with the moduli space of genus g spin curves, $\mathcal{SM}_g$,

$$(\mathfrak{M}_g)_{bos} = \mathcal{SM}_g.$$

and so $\mathcal{SM}_g \subset \mathfrak{M}_g$. This expresses the inclusion in the previous slide.

There is a forgetting map $\mathcal{SM}_g \to \mathcal{M}_g$ making $\mathcal{SM}_g$ into a finite, étale cover of $\mathcal{M}_g$, the ordinary moduli space of genus g curves. With the natural embedding $\mathcal{SM}_g \subset \mathfrak{M}_g$, we get a diagram

$\mathfrak{M}_g$ is *projected* if there exists a projection $\mathfrak{M}_g \to \mathcal{SM}_g$.

projectedness of supermoduli space

What we know about the projectedness of supermoduli space for $g \geq 2$:

- $\mathfrak{M}_g$ is not projected for $g \geq 5$. In other words, for genus $g \geq 5$, there is *no* projection $\mathfrak{M}_g \to \mathcal{SM}_g$.
- $\mathfrak{M}_2^+$ is projected. In other words, there is a projection $\mathfrak{M}_2^+ \to \mathcal{SM}_2^+$. In fact, this projection comes from the super period matrix which appears in DHP's definition of the supermeasure.
- Theorem (Donagi, O.): $\mathfrak{M}_g$ is projected away from the so-called *bad divisor* $\mathcal{B} \subset \mathfrak{M}_g$ for all genus g.
- Theorem (Donagi, O.): $\mathfrak{M}_3^+$ is projected.

To define the bad divisor, need to recall some stuff living on supermoduli space:

- The universal curve $\pi : \mathfrak{X}_g \to \mathfrak{M}_g$ on supermoduli space. Recall that every family $X \to T$ of SRSs is a pullback of $\mathfrak{X}_g$ by a unique map $T \to \mathfrak{M}_g$.
- the relative dualizing sheaf $\omega := \mathrm{Ber}(\mathfrak{X}_g/\mathfrak{M}_g)$ on $\mathfrak{X}_g$, this thing is the super version of the canonical bundle—we mostly care about $\pi_*\omega$.
- The fiber of $\pi_*\omega$ at a point X (a SRS) in $\mathfrak{M}_g$ is isomorphic to $H^0(X, \mathrm{Ber}(X))$.
- and there is an identification

$$H^0(X, \mathrm{Ber}(X)) = H^0(C, \Omega^1_C) \oplus \Pi H^0(C, L)$$

where (C, L) is the spin curve determined by X.

oh no!: $\pi_*\omega$ is not a vector bundle on $\mathfrak{M}_g$:...

Why is $\pi_*\omega$ not a vector bundle ?

Grauert theorem (in general): A coherent sheaf $\mathcal{F}$ on Y is locally free if and only if the dimension of the fibers $\gamma(y) = \dim \mathcal{F}|_y$ is constant on Y.

Applying to $\pi_*\omega$: the dimension of the fiber $\pi_*\omega|_X$ at a point X in $\mathfrak{M}_g$ is the dimension of the super vector space $H^0(C, \Omega^1_C) \oplus \Pi H^0(C, L)$. Generically, $H^0(C, L) = 0$, so $\pi_*\omega$ is not locally free over the locus in $\mathfrak{M}_g$ where $h^0(L) > 0$. This locus is a divisor, called the *bad divisor* and denoted by $\mathcal{B}$.

We let $U_g \subset \mathfrak{M}_g$ denote the complement of the bad divisor $\mathcal{B}$.

We assume $g \geq 2$.

Theorem (Donagi, O,)

$\mathfrak{M}_g$ is projected away from the so-called bad divisor $\mathcal{B} \subset \mathfrak{M}_g$ for all genus g, i.e., U_g is projected for all genus g.

Proof sketch.

- Main point: every smooth and affine superscheme is projected (and also split). So we will show that U_g is smooth and affine. The only non-trivial part is affineness.
- From AG: The complement of a very ample (positive) divisor is affine. So we will show that $\mathcal{B}$ is very ample. Suffices to show that the bosonic reduction $\mathcal{B}_{bos}$ in $\mathcal{SM}_g$ is very ample.
- Classical geometry: $\mathcal{B}_{bos}$ is the theta-null divisor in $\mathcal{SM}_g$, and this is known to be very ample (still looking for a reference...)

$\square$

The measure for the ordinary scattering amplitude is defined in two steps:

- Mumford isomorphism: $\mu_g : \det(\pi_* \omega_{\mathcal{C}_g/\mathcal{M}_g})^{13} \cong \omega_{\mathcal{M}_g}$. Set $\Omega := \pi_* \omega_{\mathcal{C}_g/\mathcal{M}_g}$.
- We make $\mu_g \otimes \overline{\mu}_g$ into a volume form on $\mathcal{M}_g$ by trivializing Ω with the Hermitian metric $\frac{i}{2} \int_C w \wedge \overline{v}$ and then taking determinant and products.
- Side remark: This metric can be expressed as the imaginary part of the period matrix .

Supermeasure: We have a super Mumford isomorphism

$$\mu_g : \mathrm{Ber}(\pi_* \omega)^5 \cong \mathrm{Ber}(\mathfrak{M}_g)$$

where recall $\pi_* \omega = \pi_* \mathrm{Ber}(\mathfrak{X}_g / \mathfrak{M}_g)$.

Problem: (1) $\pi_* \omega$ is not locally free, and (2) there is no analog of $\int_C w \wedge \overline{v}$

Resolution: Work away from the bad divisor, so on U_g where $\pi_* \omega$ is locally free, and then go directly to expressing the hermitian metric in terms of the super period matrix.

Super period map and projectedness

Next, we will investigate a connection between the super period map and the projectedness. We will do the following:

- Define periods on super Riemann surfaces, called super period, and define the super period map for a single super Riemann surface, and then extend the definition to families of super Riemann surfaces, and in particular for the universal family $\pi : \mathfrak{X}_g \to \mathfrak{M}_g$.
- Define the super period map (better would be super Torelli map), from first principles, and find that its domain of definition is U_g.

Our main result is:

Theorem

In genus $g = 2$ and $g = 3$, the super period map defines a projection $U_g \to U_{g,bos}$.

Ordinary Periods

Let C be a curve (Riemann surface).
For any global section ω of $H^0(C, \Omega^1_C)$, integration along loops determines a map $\pi_1(C) \to \mathbb{C}$. Since $\mathbb{C}$ is abelian, this map determines an element in $H^1(C, \mathbb{C})$, and we get a map

$$H^0(C, \Omega^1_C) \to H^1(C, \mathbb{C})$$

called the period map for C.

From the derived exponential sequence and Serre duality the period map can be defined for families of curves, and, in particular, for the universal family $\pi : \mathcal{C}_g \to \mathcal{M}_g$ on moduli space,

$$\pi_* \Omega^1_{\mathcal{C}_g/\mathcal{M}_g} \to R^1 \pi_* \mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{M}_g}$$

The fiber of this map at a point C in $\mathcal{M}_g$ is the period map for C.

Super periods

Let X be a super Riemann surface. A global section ω of $\mathrm{Ber}(X)$ can be integrated over any smooth submanifold of X of real codimension $1|0$, called *integration cycles*. Every loop α on the ordinary curve C underlying X can be *thickened* into an integration cycle $\tilde{\alpha}$ on X, in a way that is unique up to homology.

By integrating along thickenings of loops, any global section ω of $\mathrm{Ber}(X)$ gives rise to a map $\pi_1(X) = \pi_1(C) \to \mathbb{C}$. This map determines an element in $H^1(X, \mathbb{C}) = H^1(C, \mathbb{C})$ since $\mathbb{C}$ is abelian, and so we get a map

$$\Omega(X) : H^0(X, \mathrm{Ber}(X)) \to H^1(X, \mathbb{C}) \tag{1}$$

called the *super period map for X*.

The super period map for a super Riemann surface X in U_g is equal to the period map for $C = X_{bos}$.

Super periods cont.

We can define the super period map for a family of super Riemann surface, and, in particular, for the universal curve $\pi : \mathfrak{X}_g \to \mathfrak{M}_g$ using the derived exponential sequence and super Serre duality,

$$\pi_*\omega \to R^1\pi_*\mathbb{Z} \otimes \mathcal{O}_{\mathfrak{M}_g}$$

where $\omega := \mathrm{Ber}(\mathfrak{X}_g/\mathfrak{M}_g)$.

But this is not a morphism of vector bundles because $\pi_*\omega$ is not locally free, so we restrict to U_g, and let

$$\pi : X_g = \mathfrak{X}_g \times_{\mathfrak{M}_g} U_g \to U_g$$

so that we get a morphism of vector bundles,

$$\pi_*\omega|_{U_g} \to R^1\pi_*\mathbb{Z} \otimes \mathcal{O}_{U_g}$$

and $\omega := \omega$

Ordinary torelli map: a morphism $\mathcal{M}_g \to \mathcal{A}_g$ sending a curve C to its Jaocbian $J(C)$, which is a ppav of dimension equal to the genus of C.

1. Recall period map for the universal curve over ordinary moduli space,

$$\pi_*\omega \to R^1\pi_*\mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{M}_g},$$

2. Locally, on $\mathcal{M}_g$, choose symplectic $\phi : R^1\pi_*\mathbb{Z} \cong \mathbb{Z}^{2g}$, and get a map

$$\pi_*\omega \to \mathcal{O}^{\oplus 2g}$$

   that determines a map $\mathcal{M}_g \to \mathrm{Grass}(g, 2g)$ depending on ϕ.
3. We go to $\tilde{\mathcal{M}}_g \to \mathcal{M}_g$ parameterzing ϕs, and there get a map

$$\pi_*\tilde{\omega} \to R^1\pi_*\mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_{\tilde{\mathcal{M}}_g}$$

   that dermines a morphism $\tilde{\mathcal{M}}_g \to \mathrm{Grass}(g, 2g)$.
4. the Riemann bilinear relations then tell us that this map factors through $\mathcal{H}_g$. Furthermore, the action of mapping class group on $\tilde{\mathcal{M}}_g$ factors through the action of $\mathrm{Sp}(2g, \mathbb{Z})$ on $\mathcal{H}_g$ to get the period map, $\mathcal{M}_g \to \mathcal{H}_g / \mathrm{Sp}(2g, \mathbb{Z}) = \mathcal{A}_g$.

Properties of the ordinary torelli map $\mathcal{M}_g \to ag$:

- is an isomorphism for $g = 2$ and $g = 3$,
- for $g \geq 4$, it is an embedding away from the hyperelliptic locus in $\mathcal{M}_g$.

The Schottky locus J_g is the image of the period map. It has the following properties,

- $J_g = \mathcal{A}_g$ for $g = 2, 3$.
- In $g = 4$, J_g is a divisor in $\mathcal{A}_g$, cut out by one equation, a modular form of weight 8.
- Open problem: Find all equations for the Schottky locus in higher genus.

Unlike the ordinary period map, the super torelli map is only a rational morphism

$$\mathfrak{M}_g - - - - > \mathcal{A}_g,$$

with the following properties:

- Its domain of definition is U_g, the complement of the bad divisor, i.e.,

$$U_g \to \mathcal{A}_g.$$

The super Schottky locus $\mathfrak{J}_g$ is the image of the super period map:

- $\mathfrak{J}_g \supset J_g$ is a nilpotent thickening of the usual Schottky locus.
- $\mathfrak{J}_g = J_g (= \mathcal{M}_g = \mathcal{A}_g)$ if $g = 2, 3$.
- Theorem(FKP, 2021): If $g \leq 11$, then the super period map extends to a regular morphism $\mathfrak{M}_g^+ \to \mathcal{A}_g$, i.e., the extension over $\mathcal{B}^+$ is regular

Super torelli map:

1. Recall super period map for $\pi : X_g \to U_g$,
$$\pi_* \omega \to R^1 \pi_* \mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_{U_g},$$
2. Locally, on U_g, choose symplectic $\phi : R^1 \pi_* \mathbb{Z} \cong \mathbb{Z}^{2g}$, and get a map
$$\pi_* \omega \to \mathcal{O}_{U_g}^{\oplus 2g}$$
that determines a map $\mathcal{M}_g \to \mathrm{Grass}(g, 2g)$ depending on ϕ.
3. We go to $\tilde{U}_g \to U_g$ parameterzing ϕs, and there get a map
$$\pi_* \tilde{\omega} \to R^1 \pi_* \mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_{\tilde{U}_g}$$
that dermines a morphism $\tilde{U}_g \to \mathrm{Grass}(g, 2g)$.
4. Sicne the super period map restricts on each X in U_g to the ordinary period map, we can again use the Riemann bilinear relations to conclude that this map factors through $\mathcal{H}_g$. The mapping class group is the same and still factors to get the map,
$$U_g \to \mathcal{H}_g / \mathrm{Sp}(2g, \mathbb{Z}) = \mathcal{A}_g.$$

Theorem (Donagi, O.)

In genus $g = 2$ and $g = 3$, the super period map is a projection $U_g \to U_{g,bos}$

Proof.

The ordinary period map is an isomorphism in genus $g = 2$ and $g = 3$, and we can use it to identify $\mathcal{A}_g = J_g = \mathcal{M}_g$, and the torelli map with $U_g \to \mathcal{M}_g$, which we can lift to map $U_g \to U_{g,bos}$ using the fact that the forgetting map is étale. $\square$

Corollary (Donagi, O.)

In genus $g = 2$ and $g = 3$, the projection $U_g \to U_{g,bos}$ extends to a projection $\mathfrak{M}_g^+ \to \mathcal{SM}_g^+$. In particular, $\mathfrak{M}_3^+$ is projected.