Thomas-Yau conjecture backgrounds

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Special Lagrangian

- Let (X, ω, Ω) be a Kähler manifold with a nowhere vanishing holomorphic volume form. An n-dimensional submanifold (or some weaker notion, eg. integral current) is called **special Lagrangian**, if

$$\omega|_L = 0, \quad \mathrm{Im}(e^{-i\hat{\theta}}\Omega)|_L = 0.$$

Volume minimizer

- We assume the metric is Calabi-Yau. Then L is a minimal submanifold.
- In fact, any submanifold in the same homology class satisfies

$$\int_L Re(e^{-i\hat{\theta}}\Omega) \leq \int_L dvol = \text{Vol}(L),$$

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saturated precisely by special Lagrangians.
- Thus if a special Lagrangian exists then it is an **absolute volume minimizer**.

Almost calibrated Lagrangians

- Recall the **Lagrangian angle** is defined by

$$\Omega|_L = e^{i\theta} dvol_L.$$

  Here $\theta : L \to S^1$ is assumed to lift to $\mathbb{R}$ (**graded Lagrangians**).
- Special Lagrangians have constant phase angle $\theta = \hat{\theta}$.
- **Almost calibrated** means the Lagrangian angle is inside the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Thus L is automatically graded.
- **Quantitative almost calibrated** means $\theta \in (-\frac{\pi}{2} + \epsilon, \frac{\pi}{2} - \epsilon)$. It implies an apriori volume bound

$$\mathrm{Vol}(L) \leq \frac{1}{\sin \epsilon} \int_L \mathrm{Re}\Omega.$$

- However, the volume minimizer within a given homology class needs not be a special Lagrangian (Schoen, Wolfson). *'Direct minimization of volume is not good enough.'*
- The known construction techniques: high symmetry, gluing style constructions, integrable system (reduce to ODE or Riemann surface), Cartan-Kähler theory.
- Existence question is a major open problem in general.

What is Thomas-Yau conjecture?

- Thomas-Yau principle: **'The existence and uniqueness of unobstructed special Lagrangian branes should be governed by a stability condition on the (derived) Fukaya category.'**
- Thomas-Yau's main motivations: mirror analogy with stable vector bundles.
- Their main evidence: uniqueness theorem (further developed by Joyce-Imagi-Santos, Imagi, Abouzaid-Imagi).

Potential significance of the Thomas-Yau philosophy:

- Produce special Lagrangians.
- Mirror symmetry beyond homological mirror symmetry.
- (Far beyond the current technology) special Lagrangian enumerative invariants?

Caveats:

- The notion of stability is meant to be tentative in Thomas-Yau's proposal.
- The mirror version of stability is not really meant to be μ-stability for Hermitian-Yang-Mills connections. A slightly better mirror candidate is deformed Hermitian-Yang-Mills, though I expect it is also only approximate.

Joyce's update

The most significant progress since Thomas-Yau was the update by Dominic Joyce.

- Joyce says there should be a **Bridgeland stability condition** on the derived Fukaya category, such that the semistable objects of given phase $\hat{\theta}$ are represented by Lagrangian branes with arbitrarily small phase oscillation $|\theta - \hat{\theta}| \ll 1$. (Morally, represented by special Lagrangian branes, although these may be too singular).

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- Joyce says the way to construct this stability condition is to run **Lagrangian mean curvature flow** with surgery, and take the infinite time limit.
- Joyce says the role of unobstructed brane structure and the Fukaya category machinery is to rule out the worst singularities in the flow.

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- The derived Fukaya category is supposedly idempotent closed automatically.
- The subcategory generated by almost calibrated Lagrangians is supposedly the heart of a bounded t-structure, and in particular is an abelian category, and generates the entire $D^b Fuk$.
- Joyce hopes the Lagrangian mean curvature flow only encounters finitely many surgeries.

Question: Can we formulate the Thomas-Yau conjecture in a version circumventing these strong predictions?

Thomas-Yau conjecture

My attempted interpretation of Thomas-Yau:

- All Lagrangian branes involved are almost calibrated and unobstructed by assumption. They can be immersed (or perhaps more singular).
- We say L is **Thomas-Yau semistable** if for any exact triangle of almost calibrated branes

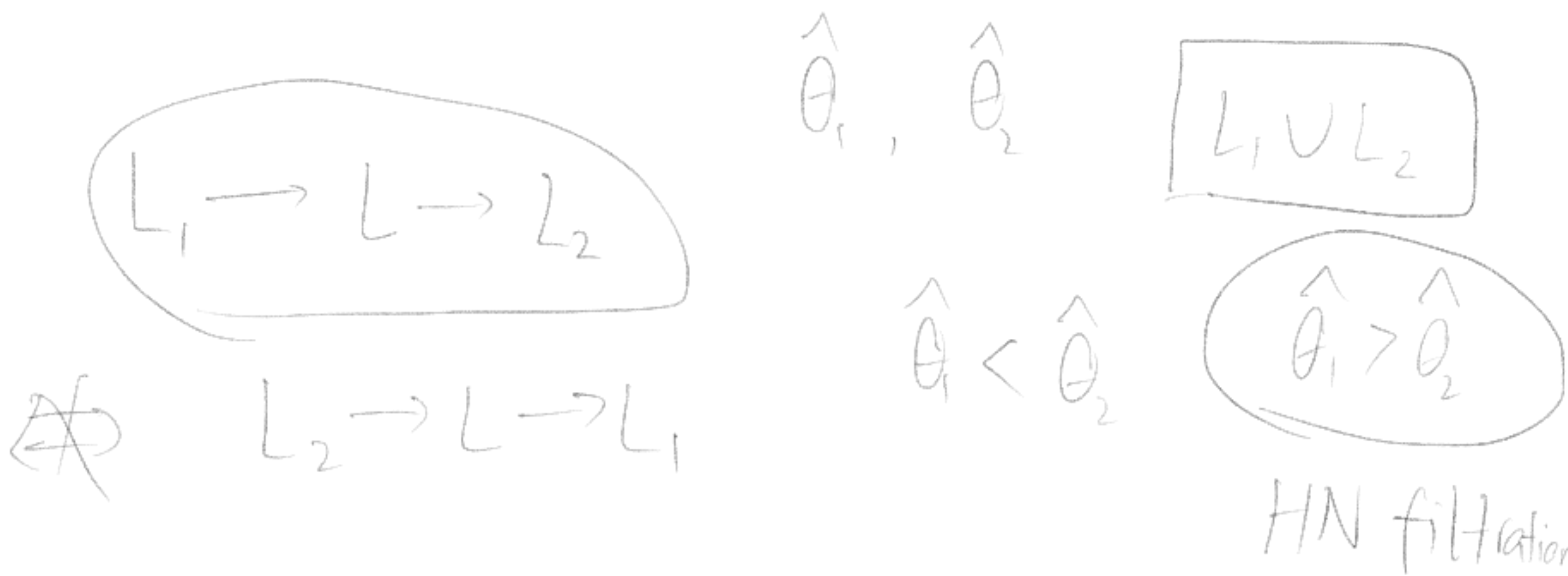
$$L_1 \to L \to L_2 \to L_1[1],$$

we have the phase angle inequality

$$\hat{\theta}_1 = \arg \int_{L_1} \Omega \leq \hat{\theta}_2 = \arg \int_{L_2} \Omega.$$

Thomas-Yau conjecture

Thomas-Yau conjecture: consider the quantitatively almost calibrated Lagrangians inside a given $D^bFuk(X)$ class, which is nonempty by assumption. There is a special Lagrangian inside the geometric measure theoretic closure, if and only if the $D^bFuk(X)$ class is Thomas-Yau semistable.



Thomas-Yau conjecture Symplectic aspects

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What are the Floer theoretic obstructions to special Lagrangians? (*i.e.* when can we rule out the existence of special Lagrangians in certain derived Fukaya category classes?)

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(time permitting) What is the functional governing the existence of special Lagrangians?

Main message: you should look at $(n-1)$-dimensional moduli spaces of holomorphic curves, and the current swept out by the $(n+1)$-dimensional family.

Setting

- We work in the exact setting. The ambient manifold is a **Stein** complex manifold $\omega = \sqrt{-1}\partial\bar{\partial}\phi$, with a nowhere vanishing holomorphic volume form Ω.
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- The Lagrangians are **exact, compact, and almost calibrated**.
- Recall exactness means

$$d\lambda = \omega, \quad df_L = \lambda|_L.$$

Caveat: we do not require f_L to take the same value at self intersections of immersed Lagrangians. There can be teardrop curves. Almost calibrated means

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Symplectic background

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- The Floer degrees at intersection points: (different from Seidel convention!)

$$\mu_{L,L'}(p) = \frac{1}{\pi}(\sum_{1}^{n} \phi_i + \theta_L(p) - \theta_{L'}(p)),$$

where

$$T_pL = \mathbb{R}^n \subset \mathbb{C}^n, \quad T_pL' = (e^{i\phi_1}, \dots e^{i\phi_n})\mathbb{R}^n.$$

Symplectic backgrounds: open-closed map

A basic ingredient for the Thomas-Yau conjecture is that the central charge function

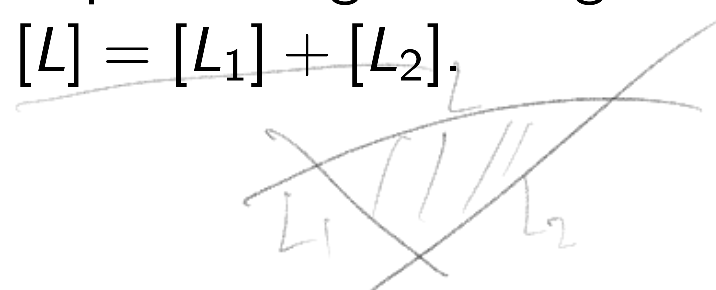
$$Z(L) = \int_L \Omega$$

needs to be well defined on the derived Fukaya category class. In fact there is a well defined map from the Grothendieck group of $D^b Fuk(X)$ to the middle homology:

$$L \mapsto [L] \in H_n(X).$$

$\partial \mathcal{C} = L - L'$

This is known to experts as a special case of the **open-closed map**. In particular, isomorphism in $D^b Fuk(X)$ implies being homologous, and exact triangle $L_1 \to L \to L_2$ implies $[L] = [L_1] + [L_2]$.



Open-closed map

Question

Given L, L' isomorphic in $D^b Fuk$, why are they homologous?

- Oversimplified answer: take the generators $\alpha \in HF^0(L, L')$, and $\beta \in HF^0(L, L')$, whose compositions are the identities. The **moduli space** of (perturbed) holomorphic curves between intersections contributing to α, β are $(n-1)$-dimensional, so the **universal family** of these curves gives rise to an $(n+1)$-dimensional integration current $\mathcal{C}$. Its boundary is $L - L'$.

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- More accurately, one needs to take into account the bounding cochain data, and the difference between cohomological units and geometric units.

Two assumptions

- **Automatic transversality assumption**: all the holomorphic curves (no perturbation!) involved in the construction of the 'bordism current' $\mathcal{C}$ are smooth points of the moduli space.

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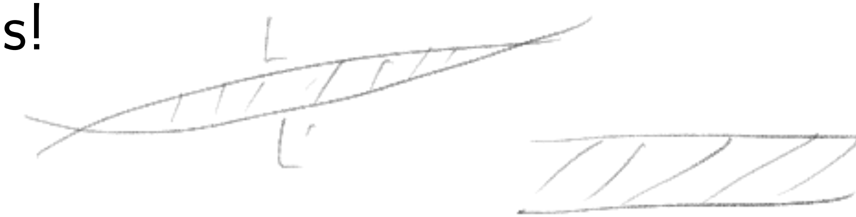
- Generally speaking, there are many $(n-1)$-dim moduli spaces corresponding to the many Lagrangian intersection points in the HF^0 generators α, β. The moduli spaces come with orientations, and upon the evaluation of $\partial\Sigma \to L \cup L'$, we can compare this orientation with the orientation of L and L'.
- **Positivity condition**: all holomorphic curves contribute to $\partial\mathcal{C} = L - L'$ with the same orientation sign.

Morse theory analogy

In **Morse theory**, the fundamental class of a compact oriented manifold L can be viewed as follows:

- The generators of the zeroth and the n-th Morse cohomology are given by the sum of local maxima/minima.
- The fundamental cycle $[L] \in H_n(L)$ is the integration current swept out by the union of the $(n-1)$-dim moduli space of gradient flowlines between local maxima and local minima.
- Notice at each generic point on L, there is only one gradient flowline passing through. We *do not have cancellation* of $\pm$ oriented flowline contributions!



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Question

Is there a general criterion for the positivity condition in the Floer theory setting, eg. assuming almost calibrated Lagrangians etc?

A priori expectation for obstructions

B-side

$\int_X$ $\quad X = \{ \text{maps pts} \to X \}$

Recall the short proof why Hermitian-Yang-Mills implies slope semistability:

- Curvature decreases in holomorphic subbundles-> HYM connection leads to a pointwise inequality.
- Integrate over the Kähler manifold to derive a global Chern number inequality-> slope stability.

A-side

$\mathcal{M} = \{ \text{holo curves } \Sigma \to X \}$

$\left(\int_{\mathcal{M}} (\quad) \right) \to$ sign

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- The inputs from Floer theory are **exact triangles** in the derived Fukaya category.

$$0 \to E_1 \to E \to E_2 \to 0$$

$$L_1 \to L \to L_2 \xrightarrow{[1]} L_1$$

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- The inputs from Floer theory are **exact triangles** in the derived Fukaya category.
- The role of holomorphic volume forms enters via **cohomological integrals**.

Looking for obstructions

Concretely: given an **exact triangle** $L_1 \to L \to L_2$ of exact, almost calibrated, compact, unobstructed Lagrangians. **Destabilizing condition**:

$$\hat{\theta}_1 = \arg \int_{L_1} \Omega > \hat{\theta}_2 = \arg \int_{L_2} \Omega.$$

Question

Does the existence of a destabilizing exact triangle rule out the possibility of L being a special Lagrangian?

An unsatisfactory answer: if we know L_1, L_2 are represented by special Lagrangians of phase $\hat{\theta}_1, \hat{\theta}_2$, then

$$\sup_L \theta \geq \hat{\theta}_1, \quad \inf_L \theta \leq \hat{\theta}_2.$$

$\hat{\theta}_1 > \hat{\theta}_2$ $\Rightarrow$ L can't be Slag.

- Reason: if $\sup_L \theta < \hat{\theta}_1$, the formula for the Floer degrees of Lagrangian intersection points implies $CF^0(L_1, L) = 0$. Thus the holomorphic curves contributing to the bordism current $\mathcal{C}$ with $\partial\mathcal{C} = L - L_1 - L_2$ cannot pass from L_1 to L. Any curve passing through L_1 is stuck on L_1, which is impossible for almost calibrated Lagrangians due to the absence of $CF^{-1}(L_1, L_1)$ intersection points.

$HF^0(L_1, L) = 0.$

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- This answer is unsatisfactory because when we are looking for special Lagrangians, we are not supposed to assume the existence of any special Lagrangians.
- If one believes in Joyce's picture of Bridgeland stability, then one can consider the Harder-Narasimhan filtration of L_1, L_2, and a version of the above arguments suggests the existence of the destabilizing exact triangle indeed rules out L being special Lagrangian.

Looking for obstructions

$L_1 \to L \to L_2$

$\hat{\theta}_1 > \hat{\theta}_2$ $\qquad$ $\hat{\theta}_1 = \arg \int_{L_1} \Omega$

Question

Can destabilizing exact triangles obstruct the existence of special Lagrangians without Lagrangian angle assumptions on L_1, L_2 beyond being almost calibrated?

- Answer: Yes, if we assume the **automatic transversality and the positivity condition** on the bordism current $\mathcal{C}$ between L and $L_1 + L_2$.
- Technique: **integration over moduli space**.

Theorem

Assume automatic transversality+ positivity condition+ destabilizing exact triangle. Then

$$\sup_L \theta \geq \hat{\theta}_1 > \hat{\theta}_2 \geq \inf_L \theta.$$

In particular L cannot be a special Lagrangian.

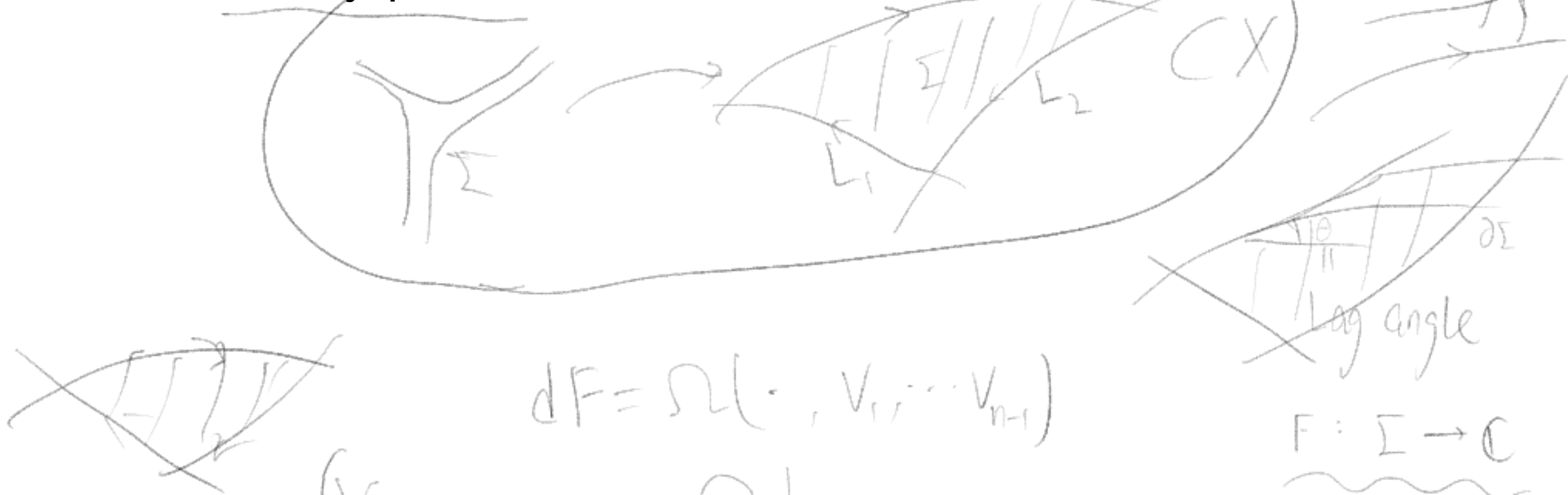
Moduli integral technique

- Strategy: express the period integrals $\int_{L_1} \Omega, \int_{L_2} \Omega$ in terms of integrals over the $n-1$ dim moduli spaces of holomorphic curves.
- Try to derive integral inequalities based on some pointwise inequality on the moduli space.

Recall some basic **deformation theory of holomorphic discs** $\Sigma \to X$ with boundary on Lagrangians (and corners at Lagrangian intersection points/bounding cochain elements):

- First order deformation vector fields are solutions to the extended linearized Cauchy-Riemann equation.
- Let $v_1, \dots v_{n-1}$ be first order deformations of holomorphic curves. Then v_i define holomorphic vector fields in the normal bundle of the image of Σ.
- The $(1,0)$-form on Σ defined by $\Omega(\cdot, v_1, \dots v_{n-1})$ is therefore holomorphic. It must be the differential of a **holomorphic function** F on the domain Σ.

- In clockwise order (in my conventions), the Lagrangian boundary of Σ encounters L, L_2, L_1. (More generally, there is a possibility to skip L_1 or L_2.)
- The function F is a holomorphic map from the disc Σ to $\mathbb{C}$. Along $\partial\Sigma$, the **incline angle** of dF is equal to the **Lagrangian angle** of the Lagrangian boundary condition.
- Feature of **almost calibrated Lagrangians+ positivity condition**: clockwise along $\partial\Sigma$, the function ReF *increases* on the L boundary portion, and *decreases* on the $L' = L_1 \cup L_2$ boundary portion.



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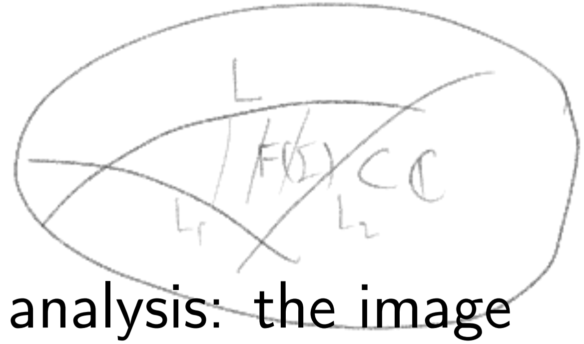


- Consequence by elementary complex analysis: the image $F(\Sigma) \subset \mathbb{C}$ lies **above** its $L_1 \cup L_2$ portion, and **below** the L portion.

Remark

(Partial justification for automatic transversality assumption) In fact, by some index computation, one can show that

- Either F is constant on Σ,
- Or $\Sigma \to X$ is a smooth point of the moduli space, and morever dF has no zero inside the interior or the boundary of Σ and only vanishes to minimal order at the corner.

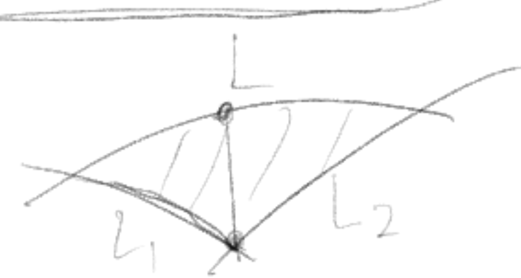
$dF = \Omega(-, v_1, \cdots v_{n-1})$

(n-1)-dim moduli

- Since the current $\partial\mathcal{C}$ sweeps out almost every point on $L - L_1 - L_2$ precisely once in the sense of counting, the period integral $\int_{L_i}\Omega$ can be expressed as an integral over the $n-1$ dim moduli space. Notice F is proportional to $v_1 \wedge \ldots \wedge v_{n-1}$, which means its proper interpretation is a family of complex valued volume forms on the moduli space.
- For each corner of Σ mapping to the $CF^1(L_2, L_1)$ point, we can find some point on the L portion of $\partial\Sigma$, with the same value of $\mathrm{Re}F$, and **bigger** value of $\mathrm{Im}F$.
- When this fact is **integrated over the moduli space**, it says that there is a subset A of L, with

$$\mathrm{Re}\int_A \Omega = \mathrm{Re}\int_{L_1}\Omega > 0, \quad \mathrm{Im}\int_A \Omega \geq \mathrm{Im}\int_{L_1}\Omega.$$

This implies $\sup_L \theta > \hat{\theta}_1 = \arg\int_{L_1}\Omega$.



Solomon functional

Jake Solomon introduced a functional among a fixed Hamiltonian isotopy class of Lagrangians, with the property that its first variation for the Hamiltonian deformation H is

$$\delta\mathcal{S} = \int_L H \mathrm{Im}(e^{-i\hat{\theta}}\Omega),$$

where $\hat{\theta} = \arg \int_L \Omega$.

Question

Can we make sense of this functional for Lagrangians inside a fixed derived category class?

Solomon functional

Answer 1: suppose L is isomorphic to L_0 in $D^b Fuk$, so in particular homologous. We take $\mathcal{C}$ so that $\partial\mathcal{C} = L - L_0$. Recall L is an exact Lagrangian with potential f_L.

$$\mathcal{S}(L) = \int_L f_L \mathrm{Im}(e^{-i\hat{\theta}}\Omega) - \int_{L_0} f_{L_0} \mathrm{Im}(e^{-i\hat{\theta}}\Omega) - \int_{\mathcal{C}} \lambda \wedge \mathrm{Im}(e^{-i\hat{\theta}}\Omega).$$

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Remark

Changing $\mathcal{C}$ by any exact integration current does not affect the functional.

Remark

This formula is more useful for the variational method, and is the starting point of the more geometric measure theoretic aspects.

Solomon functional

$$\int_\Sigma \omega + \sum_{corners} f|_-^+ = 0$$

Answer 2 (equivalent answer, under the automatic transversality assumption) The Solomon functional can be expressed as an integral over the $n-1$ dim moduli spaces $\mathcal{M}$ of holomorphic discs

$$\mathcal{S}(L) = \int_{\mathcal{M}} \mathcal{I},$$

F = const on Σ

$$\mathcal{I} = \mathrm{Im} \int_\Sigma e^{-i\hat{\theta}} F\omega + \mathrm{Im} \sum_{corners} e^{-i\hat{\theta}} Ff|_-^+,$$

where $f|_-^+$ signifies the jump in the Lagrangian potentials at the corner, and F is the holomorphic function on Σ constructed from

$$dF = \Omega(\cdot, v_1, \ldots, v_{n-1}).$$

positivity

$\Rightarrow \mathrm{Im}\, F \geq 0$

Solomon functional

Consequence of moduli space integral formula for the Solomon functional: if L_0 is a special Lagrangian, and L is almost calibrated, and assuming automatic transversality+positivity condition on the bordism current, then

$$\mathcal{S}(L) \geq \mathcal{S}(L_0).$$

- Reason: $\mathrm{Im}(e^{-i\hat{\theta}}F)$ is zero on the L_0 boundary portion of Σ, and non-negative on Σ. Morever, the bounding cochain elements on L satisfy the Novikov exponent positivity $f|_{-}^{+} \geq 0$.
- Moral: special Lagrangians should be absolute minimizers of the Solomon functional.