Extending cyclic actions to circle actions

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Geometria em Lisboa Seminar

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Question

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An effective action of a torus $T = (S^1)^r$ on (M, ω) is **Hamiltonian** if it admits a **momentum map**: a smooth map $\Phi\colon M \to \mathfrak{t}^* \cong \mathbb{R}^r$ that satisfies

$$d\Phi_j = -\iota(\xi_j)\omega$$

where $\xi_1, \ldots, \xi_r$ are the vector fields that generate the torus action.

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By the **convexity Theorem**, the image of the momentum map is a convex polytope (Atiyah, Guillemin-Sternberg, 82).

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The examples we will see today are simply connected, so every symplectic circle action is Hamiltonian.

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$\mathrm{Ham}(M, \omega)$ is the group of Hamiltonian symplectomorphisms.
A symplectomorphism φ is **Hamiltonian** if it is isotopic to the identity through a **Hamiltonian** isotopy φ_t generated $\frac{d}{dt}\varphi_t = X_t \circ (\varphi_t)$ by a family of Hamiltonian vector fields $(\iota_{X_t}\omega = dH_t)$.

The answer is No.

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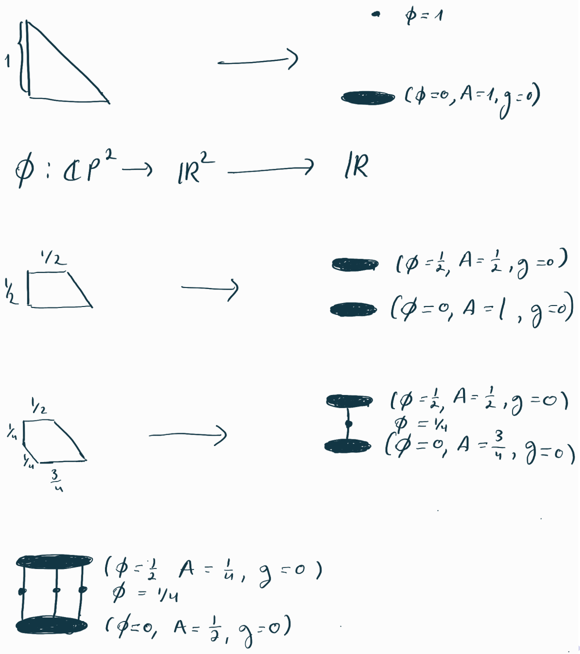
The action is Hamiltonian with a momentum map

$$\Phi([z_0 : z_1 : z_2]) = \Big(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}\Big).$$


1
$\phi = 1$
$(\phi = 0, A = 1, g = 0)$
$\phi : \mathbb{C}P^2 \to \mathbb{R}^2 \longrightarrow \mathbb{R}$
1/2
1/2
$(\phi = \frac{1}{2}, A = \frac{1}{2}, g = 0)$
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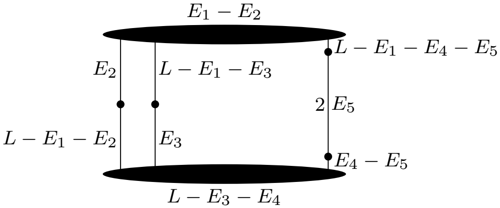
We get a circle action on $(N, \omega_N) = (\mathbb{CP}^2 \sharp 5\overline{\mathbb{CP}^2}, \omega_{1;\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{16}})$.



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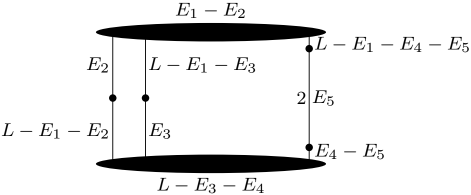

An edge indicates an invariant holomorphic sphere; the label 2 tells that the sphere in E_5 is fixed by $\mathbb{Z}^2 < S^1$; the preimage of a fat vertex is a fixed holomorphic sphere.

Perform a $\mathbb{Z}_2$-equivariant complex blowup $\pi\colon \tilde{N} \to N$ at a point p in the $\mathbb{Z}_2$-sphere in E_5 that is not S^1-fixed.

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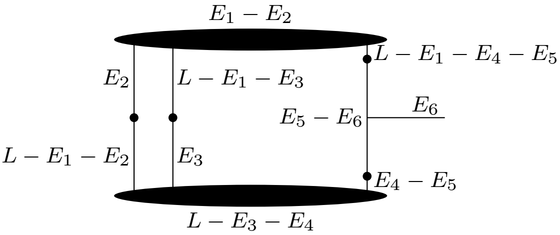

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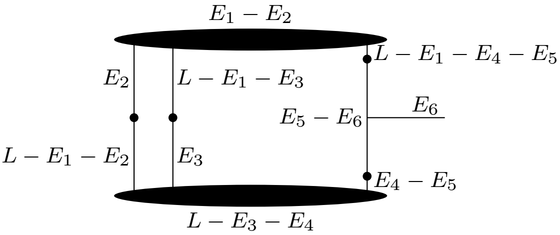

There are $\mathbb{Z}_2$-fixed spheres in $E_1 - E_2$, $L - E_3 - E_4$ and $E_5 - E_6$.

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- By Nakai's criterion, on a closed complex surface, it is enough that $\widetilde{\Omega}^2 > 0$ and $\langle \widetilde{\Omega}, [C] \rangle > 0$ for every complex curve in $\widetilde{N}$.

- The complex manifold $\widetilde{N}$ is a weak del Pezzo surface of degree 3 (obtained from $\mathbb{CP}^2$ by 6 blowups, each at a point not lying on a (-2)-curve). Hence the classes of its (-2) and (-1)-curves generate the cone of classes of complex curves, with non-negative coefficients.

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- The classes of the (-2) and (-1)-curves are the ones indicated in the configuration figure; the coupling of each with $\widetilde{\Omega}$ is positive.

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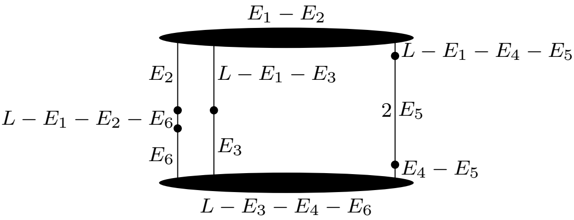
We get a homologically trivial $\mathbb{Z}_2$-action on $(\mathbb{CP}^2 \sharp 6\overline{\mathbb{CP}^2}, \omega_{1;\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{16},\frac{1}{8}})$.

We note that the manifold does admit a Hamiltonian circle action.



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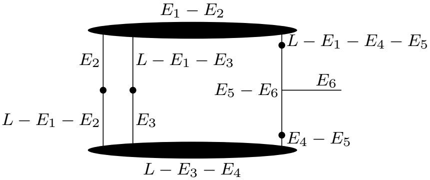
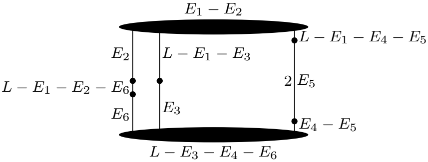
We show that the configuration of invariant holomorphic spheres in the constructed $\mathbb{Z}_2$-action is different from that of any Hamiltonian circle action on the symplectic manifold.



For example, in the action



there is a fixed surface in $L - E_3 - E_4 - E_6$. The intersection of this class with $L - E_3 - E_4$ is -1. An embedded sphere S fixed by a non-trivial subgroup H is holomorphic w.r.t. any invariant complex structure J. (For $w \in TS$ and $h \in H$, $d\sigma_h(Jw) = J(d\sigma_h w) = Jw$.) So, if the S^1-action extends the constructed $\mathbb{Z}_2$-action, we get a contradiction to the positivity of intersections of holomorphic spheres.

To go over all the circle actions on $(\mathbb{CP}^2\sharp 6\overline{\mathbb{CP}^2}, \omega_{1;\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{16},\frac{1}{8}})$, we use Karshon- K. - Pinsonnault characterization of circle actions on symplectic blowups of $\mathbb{CP}^2$: each action is obtained by S^1-equivariant symplectic blowups of sizes $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{16}, \frac{1}{8})$ from a circle action on $(\mathbb{CP}^2\sharp\overline{\mathbb{CP}^2}, \omega_{1;\frac{1}{2}})$.

(Li-Li-Wu, 22 preprint): for the form $\omega_{1;\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{16}}$ on $\mathbb{CP}^2\sharp 6\overline{\mathbb{CP}^2}$, the homologically trivial part of the symplectomorphism group is path connected. Therefore, the constructed $\mathbb{Z}_2$-action is Hamiltonian isotopic to the identity.

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So the answer to the second question is NO as well.

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The symplectic manifolds to consider are either $(\mathbb{CP}^2, \lambda\omega_{\mathsf{FS}})$ or a symplectic k-blowup of a ruled symplectic S^2-bundle over a closed Riemann surface (the symplectic form on the total space is nondegenerate on each fiber) (Karshon, 99).

Theorem

The answer is YES if (M, ω) is either $(\mathbb{CP}^2, \lambda\omega_{FS})$ (Chen, 10) or a symplectic k-blowup of a ruled symplectic S^2-bundle over S^2 in case

- $k = 0$ *(Chiang-K., 19),*
- $k = 1$ *(Chiang-K., 23 preprint),*
- $k = 2$ *and the symplectic manifold is positive monotone, i.e., $c_1(M) = r[\omega]$ for $r > 0$ (Chiang-K., 23 preprint).*

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In all these cases, any S^1-action on (M, ω) extends to a toric action.

The cyclic-to-circle extension algorithm in case all circle actions extend to toric actions.

- We find a chain of $d = \dim H_2(M)$ $\mathbb{Z}_n$-invariant embedded symplectic spheres $D_1, \ldots, D_d$.

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- We find an S^1-action on (M, ω) that extends the $\mathbb{Z}_n$-action on the chain $\cup_{i=1}^d D_i$.
- By the equivariant Weinstein's symplectic neighbourhood theorem, the $\mathbb{Z}_n$ action extends to the S^1-action on a $\mathbb{Z}_n$-equivariant neighbourhood of the chain of spheres.

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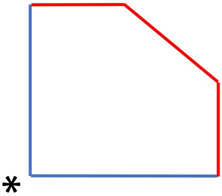

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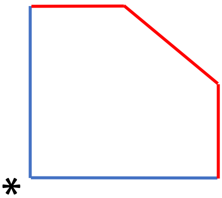

Lifting the radial vector field of $\mathbb{R}^2$ emitting from the vertex $*$ retracts the complement of the chain onto a standard T^2-invariant ball (of some size) whose momentum map image is the lower left corner.

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Therefore the $\mathbb{Z}_n$-action on (M, ω) is conjugated by a symplectomorphism to the restriction of a circle action.

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The almost complex structure J we consider is $\mathbb{Z}_n$-invariant. We use positivity of intersections to deduce that a J-holomorphic sphere in a class of negative self intersection is invariant.

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Given positive integers a, b, n such that $\gcd(a, b, n) = 1$, we need to find positive integers a', b' such that $a' \equiv a \pmod{n}$, $b' \equiv b \pmod{n}$ and $\gcd(a', b') = 1$. (We use Dirichlet prime number theorem.)

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Assume that a $\mathbb{Z}_n$-action extends to a Hamiltonian S^1-action on (M, ω).

Question

When does a $\mathbb{Z}_n$-action obtained by a $\mathbb{Z}_n$-equivariant symplectic blowup at a $\mathbb{Z}_n$-fixed point p in M extend to a Hamiltonian S^1-action?

For a fixed point p, denote by $\mathrm{Emb}^G(\epsilon, \{p\})$ the space of equivariant symplectic embeddings $i\colon (B^4(r), \omega_0) \hookrightarrow (M, \omega)$ such that $i(0) = p$ and $\epsilon = \frac{r^2}{2}$.

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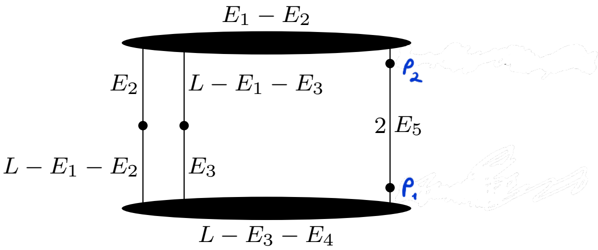
Conjecture

A $\mathbb{Z}_n$-action obtained by a $\mathbb{Z}_n$-equivariant symplectic blowup at an S^1-fixed point p of size ϵ extends to a Hamiltonian S^1-action if the inclusion $\mathrm{Emb}^{S^1}(\epsilon, M, \{p\}) \hookrightarrow \mathrm{Emb}^{\mathbb{Z}_n}(\epsilon, M, \{p\})$ induces an isomorphism on π_0.

This conjecture is consistent with our example of a $\mathbb{Z}_2$-action that does not extend to a Hamiltonian S^1-action:



At the S^1-fixed point p_i, the space $\mathrm{Emb}^{S^1}(\frac{1}{8}, \{p_i\})$ is empty but the space $\mathrm{Emb}^{\mathbb{Z}_2}(\frac{1}{8}, \{p_i\})$ is not.

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Assume that $G = S^1$ and the action is Hamiltonian or that $G = \mathbb{Z}_n$ and the action is homologically trivial. Let Σ denote a connected component of the fixed point set M^G.

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Assume that $G = S^1$ and the action is Hamiltonian or that $G = \mathbb{Z}_n$ and the action is homologically trivial. Let Σ denote a connected component of the fixed point set M^G.

Question

Is the space of equivariant symplectic embeddings $i\colon (B^4(r), \omega_0) \hookrightarrow (M, \omega)$ *such that* $i(0) \in \Sigma$ *path-connected?*

We answer this question in a current project, joint with River Chiang and Pranav Chakravarthy.