Algebraic Presentation of Cobordisms and Classification of TQFTs

joint with A. Beliakova, I. Bobtcheva, and R. Piergallini

Marco De Renzi

January 10, 2024

Algebraic Presentation of $2\mathrm{Cob}$

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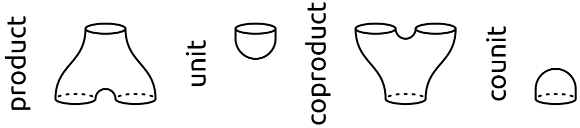
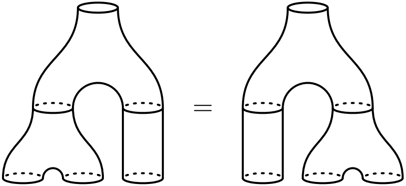
- generating object: S^1
- generating morphisms:



- relations: associativity



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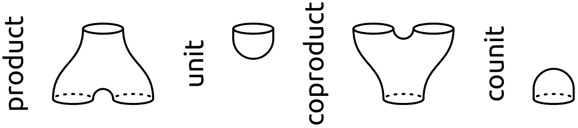

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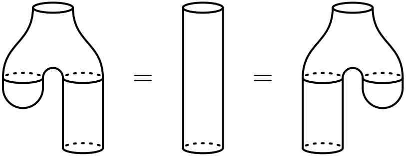

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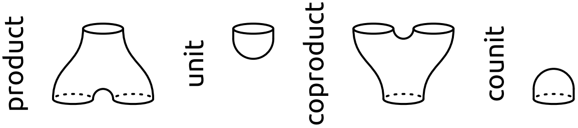

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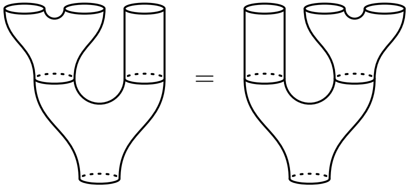

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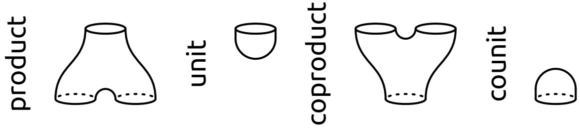

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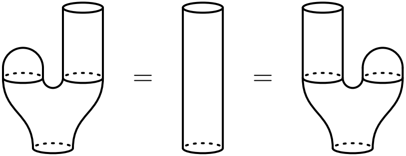

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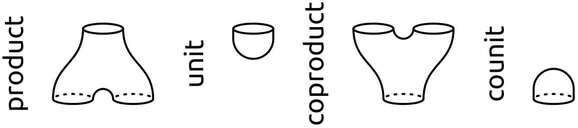

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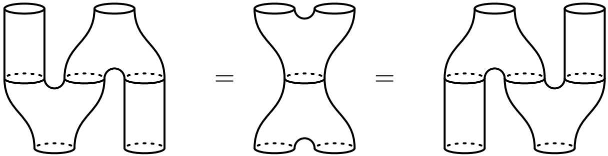

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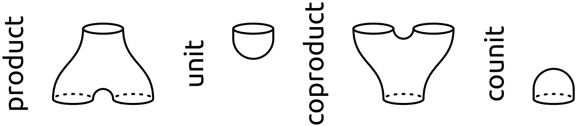

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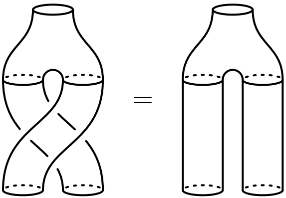

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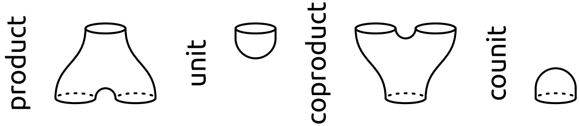

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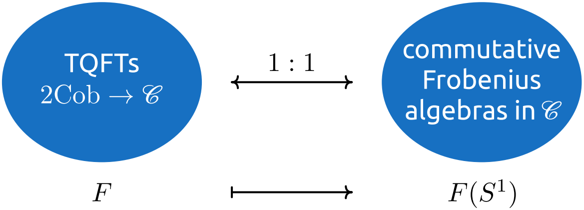
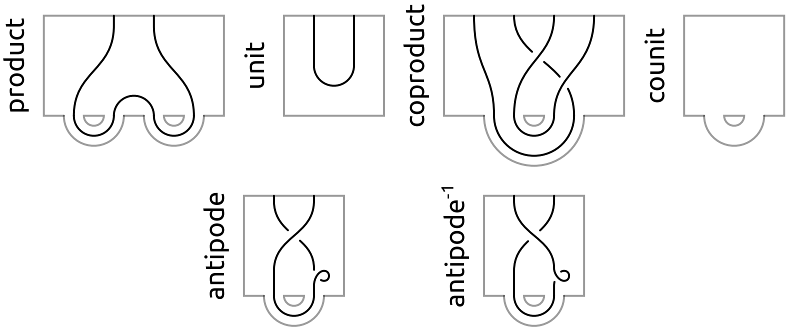


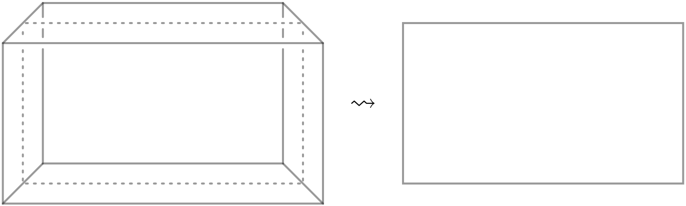
Kerler-Ohtsuki Problem

Problem (Kerler-Ohtsuki): find algebraic presentation of $3\mathrm{Cob}$ (connected cobordisms between connected surfaces with connected boundary, with $\otimes = \natural$)

Crane-Yetter: $\Sigma_{1,1} = (S^1 \times S^1) \smallsetminus D^2$ Hopf algebra in $3\mathrm{Cob}$



How To Read Diagrams



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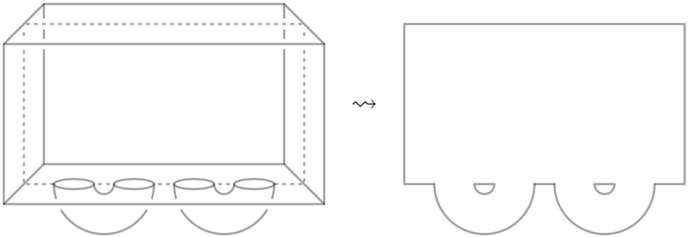

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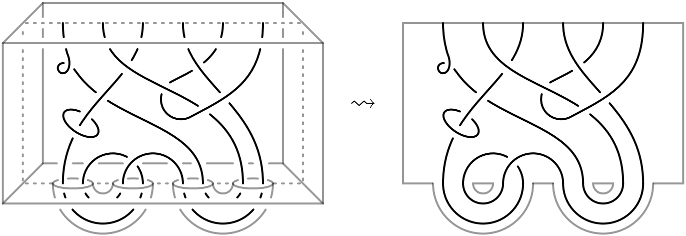

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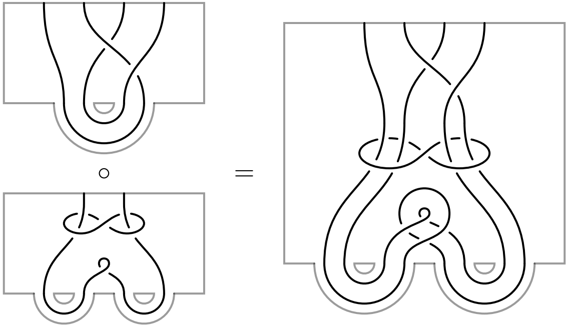

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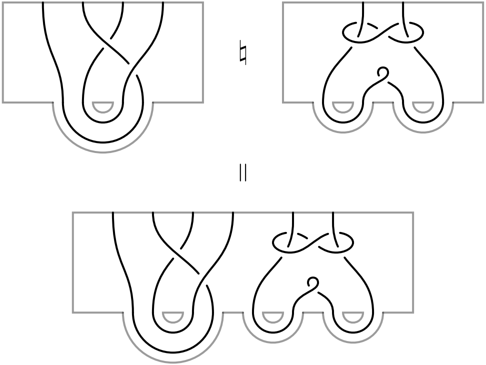
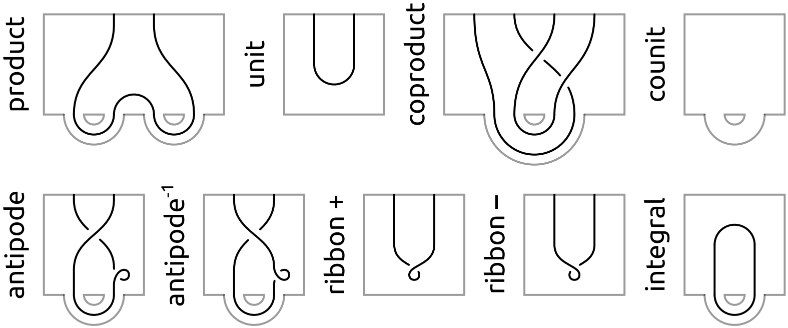


Kerler-Habiro Conjecture

Kerler: $3\mathrm{Cob}$ generated by $\Sigma_{1,1}$ and by

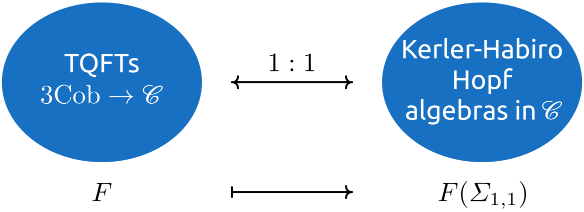


Habiro: algebraic presentation (announced without proof)

Algebraic Presentation of $\mathrm{3Cob}$

Theorem (Beliakova-Bobtcheva-D-Piergallini)

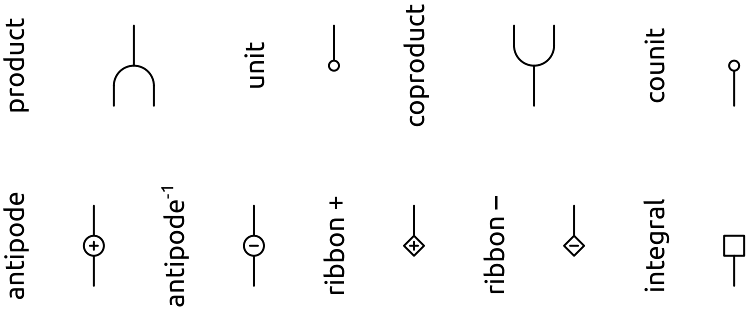
$\mathrm{3Cob}$ is the free braided monoidal category generated by a Kerler-Habiro Hopf algebra



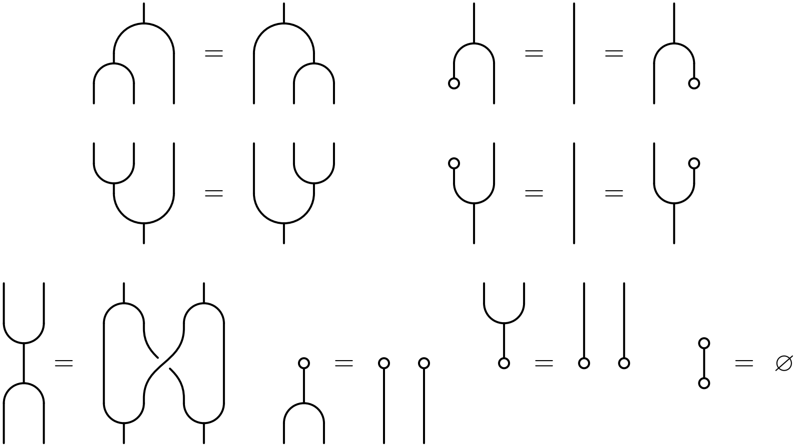
Kerler-Habiro Hopf Algebra - Structure

$\mathscr{C}$ braided monoidal category

Kerler-Habiro Hopf algebra in $\mathscr{C}$: object $\mathscr{H} \in \mathscr{C}$ equipped with structure morphisms



Kerler-Habiro Hopf Algebra - Axioms



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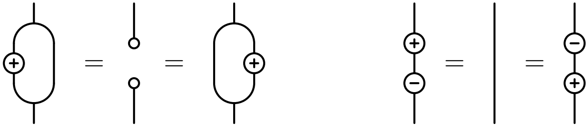

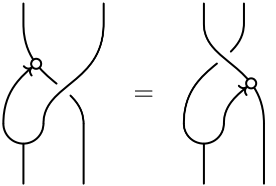

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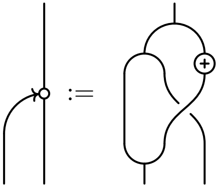

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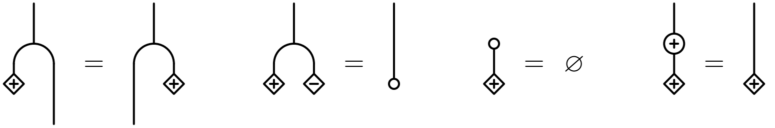

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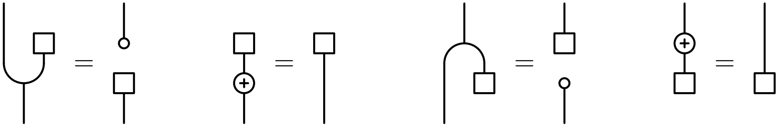

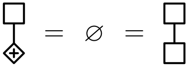

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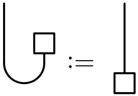

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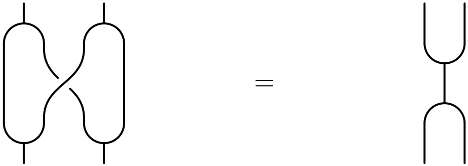

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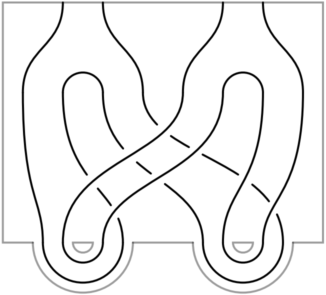

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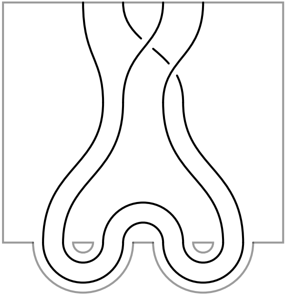

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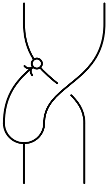

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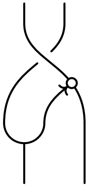

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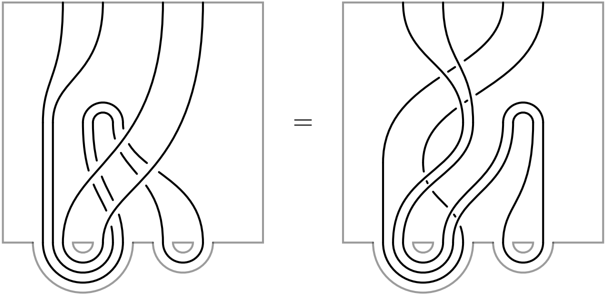
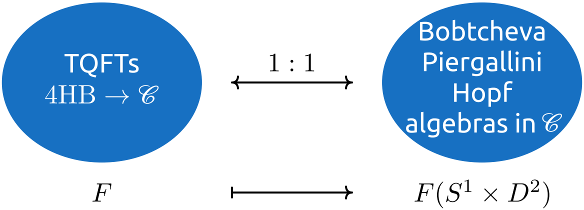


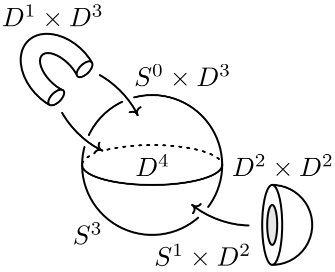
Algebraic Presentation of $\mathrm{4HB}$

Theorem (Bobtcheva-Piergallini)

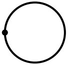
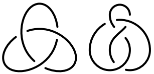
$\mathrm{4HB}$ is the free braided monoidal category generated by a Bobtcheva-Piergallini Hopf algebra



2-Handlebodies: smooth manifolds obtained from D^4 by attaching a finite number of 1-handles and 2-handles



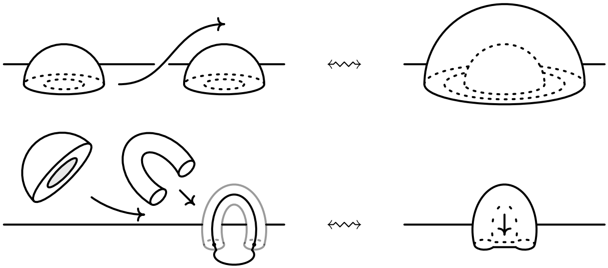
Kirby diagrams:

- 1-handles $\rightsquigarrow$ 
- 2-handles $\rightsquigarrow$  $\cdots$

3-Deformations and Diffeomorphism

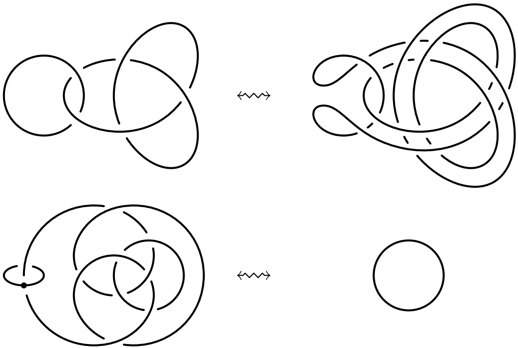
Diffeomorphic 4-dimensional 2-handlebodies are related by finite sequences of isotopies of handle attaching maps and:

- 2-handle slides
- creation/removal of canceling pairs of $1/2$-handles and of $2/3$-handles



2-Deformations and 2-Equivalence

2-Deformations: canceling pairs of $2/3$-handles are forbidden



Open problem

If two 4-dimensional 2-handlebodies are diffeomorphic, are they also 2-equivalent?

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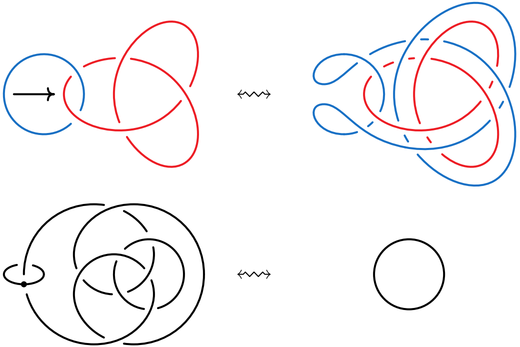

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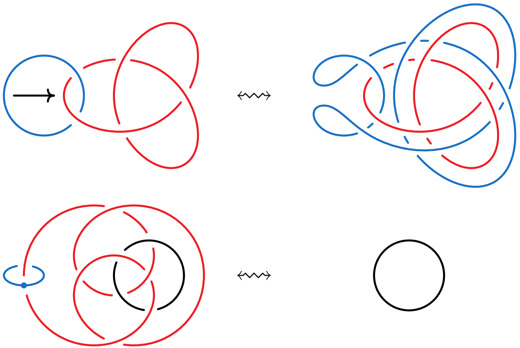

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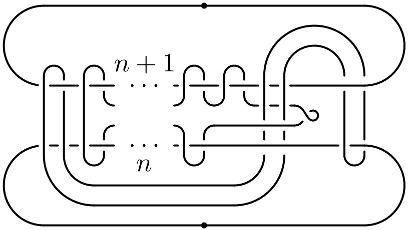
Andrews-Curtis Conjecture

Conjecture: Every 4-dimensional 2-handlebody which is diffeomorphic to D^4 is also 2-equivalent to D^4

Conjecture (Andrews-Curtis): Every balanced presentation of the trivial group can be reduced to the empty presentation by a finite sequence of Nielsen transformations

$\Delta_n =$ 

$$\pi_1(\Delta_n) = \langle x, y \mid xyx = yxy, x^n = y^{n+1} \rangle$$

Existence of TQFTs on $4\mathrm{HB}$

$4\mathrm{HB}$ category with:

- objects: 3-dimensional 1-handlebodies
- morphisms: 4-dimensional 2-handlebodies up to 2-deformations

Theorem (Beliakova-D)

Every unimodular ribbon category $\mathscr{C}$ induces a TQFT

$$J_4 : 4\mathrm{HB} \to \mathscr{C} \qquad (\text{e.g. } \mathscr{C} = H\text{-mod})$$

$$S^1 \times D^2 \mapsto \int_{X \in \mathscr{C}} X \otimes X^* \qquad (\text{e.g. } \mathrm{ad})$$

Quotient Categories

- $3\mathrm{Cob}^\sigma$ category of cobordisms with signature defects
- If $\mathscr{C}$ is factorizable, then

$$\begin{array}{ccc} 4\mathrm{HB} & \xrightarrow{J_4} & \mathscr{C} \\ & {}_{\partial_+}\searrow \quad \nearrow_{J_3} & \\ & 3\mathrm{Cob}^\sigma & \end{array}$$

- $2\mathrm{CW}$ category of 2-dimensional CW-complexes
- If $\mathscr{C}$ is unframed, then

$$\begin{array}{ccc} 4\mathrm{HB} & \xrightarrow{J_4} & \mathscr{C} \\ & {}_{\mathrm{Sp}}\searrow \quad \nearrow_{J_2} & \\ & 2\mathrm{CW} & \end{array}$$

Unimodular Ribbon Categories

$\mathscr{C}$ linear category over a field $\Bbbk$

- $\mathscr{C}$ is finite if $\mathscr{C} \cong A\text{-mod}$ for a finite-dimensional $\Bbbk$-algebra A
- $\mathscr{C}$ is ribbon if it comes equipped with
  - tensor product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$
  - tensor unit $\mathbb{1} \in \mathscr{C}$
  - duality morphisms $\overleftarrow{\mathrm{ev}}_X : X^* \otimes X \to \mathbb{1}$, $\overleftarrow{\mathrm{coev}}_X : \mathbb{1} \to X \otimes X^*$, $\overrightarrow{\mathrm{ev}}_X : X \otimes X^* \to \mathbb{1}$, $\overrightarrow{\mathrm{coev}}_X : \mathbb{1} \to X^* \otimes X \quad \forall\, X \in \mathscr{C}$
  - braiding morphisms $c_{X,Y} : X \otimes Y \to Y \otimes X \quad \forall\, X, Y \in \mathscr{C}$
  - twist morphisms $\vartheta_X : X \to X \quad \forall\, X \in \mathscr{C}$

$\mathscr{C}$ finite ribbon category

- $\mathscr{C}$ is unimodular if the projective cover $P_{\mathbb{1}}$ of $\mathbb{1}$ satisfies $P_{\mathbb{1}}^* = P_{\mathbb{1}}$
- $\mathscr{C}$ is factorizable if its Müger center $\mathrm{M}(\mathscr{C})$ is trivial
- $\mathscr{C}$ is unframed if $\vartheta = \mathrm{id}$

Instability of 2-Exotic Pairs

$$X \text{ diffeomorphic to } Y$$

$$\Downarrow$$

$$\exists\, k \geqslant 0 \text{ such that } X \natural (S^2 \times D^2)^{\natural k} \text{ 2-equivalent to } Y \natural (S^2 \times D^2)^{\natural k}$$

$$\Downarrow$$

$$J_4(X) J_4(S^2 \times D^2)^k = J_4(Y) J_4(S^2 \times D^2)^k$$

If $\mathscr{C} = H\text{-mod}$ then

- $J_4(S^1 \times D^3) \neq 0 \Leftrightarrow H$ semisimple
- $J_4(S^2 \times D^2) \neq 0 \Leftrightarrow H$ cosemisimple (i.e. H^* semisimple)