The fundamental group of ... the C^0-~~completion~~ closure! in Homeo.

$\boxed{\text{On } \pi_1(\overline{Ham}).}$

jt w/ V. Humilière & A. Jannaud

of the group of Hamiltonian diffeomorphisms.

1- Symplectic geometry.

(M, ω)

Smooth closed 2n-dim'l mfd.

non-degenerate close 2-form.

measures area of 2-dim'l stuff. (degenerate).

Exple. _ $\mathbb{R}^{2n}$, $\sum_{i=1}^{n} dx_i \wedge dy_i$ + Darboux ↳ local model.

- Σ_g : orientable surface of genus g. + ω : area form.
- T^*M for any smooth mfd M, + $\omega = d(\underbrace{-pdq}_{=\lambda})$ Liouville 1-form. $q \in M$, $p \in$ fibre.
- $\mathbb{C}P^n$, ω_{FS}

+ blow-ups
+ products (...)

$\underline{Ham(M,\omega)}$?

ω non-degenerate : $T^*M \overset{\omega^{\flat/\#}}{\longleftrightarrow} TM$.

$H : [0,1] \times M \xrightarrow{C^\infty} \mathbb{R}$: $dH_t = -\omega(X_H^t, -)$

Hamiltonian function

Hamiltonian vector field generated by H

$\hookrightarrow \phi_H^t$: $\phi_H^0 = id$ + $\frac{\partial}{\partial t}\phi_H^t = X_H^t(\phi_H^t)$.

Hamiltonian isotopy

Hamiltonian diffeo

$Ham(M,\omega) = \{ \varphi \in Diff(M) \mid \exists H, \phi_H^1 = \varphi \}$.

<u>Rmk.</u> $Ham \triangleleft Symp < Diff$

$\{ \varphi \in Diff \text{ s.t. } \varphi^*\omega = \omega \}$.

C^1 condition.

2 - C^0 Symplectic geometry.

"Once upon a time, Gromov ..." .

g : Riemannian metric on M.

$\rightsquigarrow$ d distance on M

$\rightsquigarrow$ C^0-distance on $\mathrm{Diff}(M)$:

$$d_{C^0}(\varphi, \psi) = \max_{x \in M} d(\varphi(x), \psi(x)) .$$

Thm. (Gromov) C^0-closure of Symp in Diff^{vol}

is either Symp or Diff^{vol}.

(& Eliashberg) rigidity flexibility.

Corollary. $(\psi_n)_n \subset$ Symp s.t. $\psi_n \xrightarrow{C^0} \psi$.

If ψ is smooth then $\psi \in$ Symp.

Notion of symplectic homeomorphism $\in \overline{\text{Symp}}$.

In this talk: Hamiltonian homeomorphism $\in \overline{\text{Ham}}$.

(C^0-closure in Homeo).

3. Main results.

Study of $i : \mathrm{Ham} \hookrightarrow \overline{\mathrm{Ham}}$.

- Not much known!
- Except for surfaces, i: homotopy equiv.
  (Le Calvez '05 - ...)
- Some results on $\overline{\mathrm{Ham}}$
  (Buhovsky - Humilière - Seyfaddini '18, '21)
- Some results on $i : \mathrm{Symp} \hookrightarrow \overline{\mathrm{Symp}}$.
  (Jannaud, '21, '22).

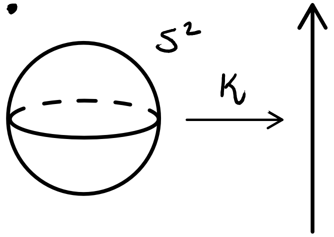
Upshot. Have a sufficient condition to ensure that certain essential loops in Ham survive in $\pi_1(\overline{\mathrm{Ham}})$.

+ several natural situations where this applies.

Thm A (HJL) Λ Hamiltonian circle action on (M,ω) rational.

Assume Λ has semi-free extremal fixed point components.

Then $i_*([\Lambda]) \neq 0 \in \pi_1(\widetilde{\mathrm{Ham}})$.

- 

K autonomous

$\rightsquigarrow \phi_K^t = \Lambda_t$ rotation.

$F_{\substack{max \\ min}} = K^{-1}\left(\substack{max \\ min}(K)\right)$

- semi-free: in a nbhd of $F_{\substack{max \\ min}}$, each pt has stabilizer $\underbrace{\{id\}}_{\text{free}}$ or $\underbrace{\Lambda}_{\text{semi-}}$.

  free + semi-

- rational: $\omega(\underbrace{\pi_2(M)}))$ discrete

  "group of periods".

Corollary. $(M,\omega) = (S^2\times S^2, \omega_1)$ or $(\mathbb{CP}^2, \omega_{FS})$

$i_* : \pi_1(\mathrm{Ham}) \to \pi_1(\widetilde{\mathrm{Ham}})$ injective.

Higher order examples?

$\boxed{\mathbb{C}P^n}$ (Seidel, '00) $\exists h_n$ of order $n+1$ in $\pi_1(\mathrm{Ham})$

Thm B. $i_*(h_n)$ of order $n+1$ in $\pi_1(\overline{\mathrm{Ham}})$.

Rmk. (Sasao '74) h_n of order $n+1$ in $\pi_1(\mathrm{Homeo})$.

$\boxed{\text{Hirzebruch}}$ (Abreu-McDuff, '00)

- $\pi_1(\mathrm{Ham}(S^2\times S^2, \omega_{\mu>1})) = \langle \underbrace{\Lambda_1}_{\text{order } 2}, \Lambda_2, \underbrace{\Lambda}_{\text{order } \infty} \rangle$
- $\pi_1(\mathrm{Ham}(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega'_\mu)) = \langle \Lambda' \rangle$. $\llcorner$ order ∞.

Thm C. $\forall \mu \in \mathbb{Q}$, injectivity of

$$i_* : \pi_1(\mathrm{Ham}(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, \omega'_\mu)) \to \pi_1(\overline{\mathrm{Ham}})$$

Rmk. (i) ω'_μ : $\mu = \text{area}(\text{blow-up})$,
$1 = \text{area}(\text{fibre})$.

(ii) $\mu \in \mathbb{Q} \leadsto (M, \omega'_\mu)$ rational.

(iii) What about $S^2 \times S^2$?!

$i_*(\Lambda) \neq 0$ ✓ $\quad i_*(\Lambda^2)$?!

4. Proof (w/ black boxes).

- $\pi_1(\mathrm{Ham}) \xrightarrow{S} QH_*(M,\omega)^\times \xrightarrow{\nu} \mathbb{R}$

Seidel's morphism — Quantum homology. — valuation of QH_*

$$QH_*(M,\omega) = H_*(M;\mathbb{k}) \otimes_{\mathbb{k}} \Lambda$$

w/ $\Lambda = \Lambda^{univ}[q,q^{-1}]$, $\mathrm{degree}(q) = 2$,

$$\Lambda^{univ} = \Big\{ \sum_{\kappa\in\mathbb{R}} r_\kappa t^\kappa \;\Big|\; r_\kappa \in \mathbb{k}, \text{ and } \forall c\in\mathbb{R},\ \#\{\kappa > c \mid r_\kappa \neq 0\} < \infty \Big\}.$$

$$\nu\Big(\sum a_\kappa \otimes q^{d_\kappa} t^\kappa\Big) = \max\{\kappa \mid a_\kappa \neq 0\}.$$

morally, d_κ remembers $\langle c_1(TM,J), \pi_2(M)\rangle$,

and κ (in t^κ) : $\langle \omega, \pi_2(M)\rangle$.

- $\Gamma : \pi_1(\mathrm{Ham}) \longrightarrow \mathbb{R}$

$$h \longmapsto \nu(S(h)) + \nu(S(h^{-1})).$$

Thm D .
$$\begin{tikzcd} \pi_1(\mathrm{Ham}) \arrow[rr, "i_*"] \arrow[dr, "\Gamma"'] & & \pi_1(\overline{\mathrm{Ham}}) \arrow[dl, "\exists \overline{\Gamma}"] \\ & \mathbb{R} & \end{tikzcd}$$

$\nu(S(id)) = \nu([M]) = 0$.

Corollary. $\underbrace{\Gamma(h)}_{\in\mathbb{R}} \neq 0 \Rightarrow \underbrace{i_*(h)}_{\in \pi_1(\overline{Ham})} \neq 0$ ✓

Proof (Thm D).

(i) $\Gamma = \tilde{\gamma}$ on $\pi_1(Ham)$ — spectral norm.

$\tilde{\gamma}$ spectral pseudo-norm on $\widetilde{Ham}$ — Viterbo '90 + Oh '98, Schwarz '00.

$\pi_1(Ham) \subset \widetilde{Ham}$.

(ii) Thm (Kawamoto '22). $\tilde{\gamma}$ is C^0-continuous up to the (discrete) group of periods.

i.e. $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $d_{C^0}(id, \phi) < \varepsilon$,

$\forall \tilde{\phi}$ which lifts ϕ to $\widetilde{Ham}$, $\exists \ell \in \mathbb{Z}$ s.t.

$$|\tilde{\gamma}(\tilde{\phi}) - \ell \cdot \Omega| < \varepsilon.$$

$\Omega > 0$ s.t. $\omega(\pi_2(M)) = \Omega\mathbb{Z}$.

$\hookrightarrow$ $\tilde{\gamma}$ descends to $Ham \xrightarrow{\gamma} \mathbb{R}/\Omega\mathbb{Z}$ C^0-cont.

$\hookrightarrow$ factors through $Ham \xrightarrow{i} \overline{Ham} \to \mathbb{R}/\Omega\mathbb{Z}$

+ π_1-functor: $\pi_1(Ham) \longrightarrow \Omega\mathbb{Z}$, $\pi_1(Ham) \to \pi_1(\overline{Ham}) \to \Omega\mathbb{Z}$ ✓

5. Computations of Γ.

- $\mathbb{CP}^n$ (Entov–Polterovich).
- Hirzebruch (McDuff–Tolman '06 $\leadsto$ Thm A)
  (Ostrover '06), (Anjos–L. '17, '18).

<u>Rmk.</u> (A-L 17) shows that $\mathcal{S}$ is injective for all Hirzebruch surfaces.

(HJL) shows that $\nu \circ \mathcal{S}$ is also injective.

<u>BUT</u> $\nu \circ \mathcal{S}$ does not factor through i.

For $(S^2 \times S^2, \omega_{\mu > 1})$, $\forall p$, $\nu(\mathcal{S}(\Lambda^p)) \neq 0$.

but $\Gamma(\Lambda^p) = 0 \iff p = 2k$.

$\Rightarrow$ <u>Question</u>. $i_*(\Lambda)$ of order 2 in $\pi_1(\overline{\mathrm{Ham}})$?

<u>Rmk.</u> About $\tilde{\gamma}$:

- $\tilde{\gamma}$ is a PSEUDO-norm on $(S^2 \times S^2, \omega_{\mu > 1})$.
- $\tilde{\gamma}$ is non-degenerate on
  $(\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2, \omega_\mu')$, $\forall \mu \in \mathbb{Q}$.

More precisely :

	$\nu(u^{-p})$	$\nu(u^{p})$
$0<\mu\leqslant\frac{1}{2}$	$(p-2\lfloor\frac{p-1}{3}\rfloor)\mu+(\lfloor\frac{p-1}{3}\rfloor+1)$	$-(p-2)\mu$
$\frac{1}{2}<\mu\leqslant 1$	$(p-2\lfloor\frac{p+2}{4}\rfloor)\mu+(\lfloor\frac{p+2}{4}\rfloor+1)$	$-(p-2\lfloor\frac{p+1}{4}\rfloor)\mu-(\lfloor\frac{p+1}{4}\rfloor-1)$
$1<\mu$	$\lfloor\frac{p+1}{2}\rfloor\mu+(\lfloor\frac{p}{4}\rfloor+1)$	$-\lfloor\frac{p}{2}\rfloor\mu-\lfloor\frac{p-1}{4}\rfloor$

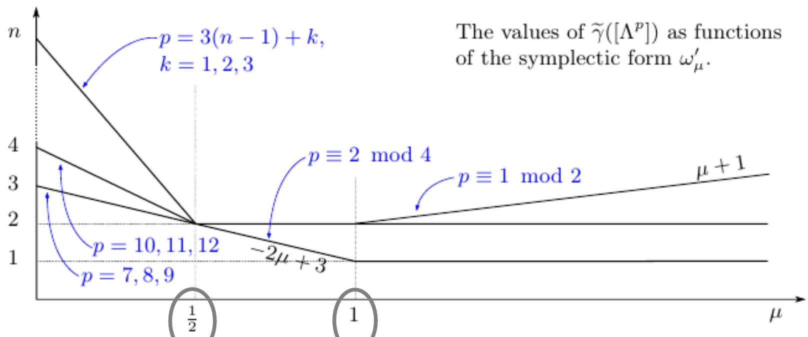
∞ number of values of $\tilde{\gamma}(\pi_1(\mathrm{Ham}))$

finite number of values of $\tilde{\gamma}(\pi_1(\mathrm{Ham}))$.

monotone $\tilde{\gamma}\leq 2$.

small blow-ups

big blow-ups



geometrically relevant monotone case!

Algebraically relevant area(fibre) = area(blow-up).

Rmk (Ostrover '06).

$\mu > 1/2$, QH_4 = field ; $\mu < 1/2$, $QH_4 = \bigoplus$ fields

+ on each summand $\nu(a)+\nu(a^{-1}) < C$ (uniform).

$\Rightarrow$ since $\tilde{\gamma}$ unbounded for $\mu < 1/2$, show that $S(h^p)$ and $S(h^{-p})$ do not belong to same summand.