Filtrations of cohomology rings from Floer theory of contracting $\mathbb{C}^*$-actions

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Very brief intro to Floer theory

- The central branch of symplectic topology is **Floer theory**.
- It studies Hamiltonian flows ("symplectic gradient") on symplectic manifolds (Y, ω).
- Given a Hamiltonian $H : Y \to \mathbb{R}$, its Floer chain complex $CF^*(H)$ is generated by Hamiltonian orbits ($\dot{x} = X_H(x)$) and differential is given by X_H-perturbed pseudoholomorphic cylinders ($\partial_s u + I(\partial_t u - X_H) = 0$).
- **Upshot:** One gets homology theory, which for closed manifolds recovers the singular homology $HF^*(H) \cong H^*(Y)$
- For open manifolds Y and a "small" Hamiltonian still get $HF^*(H) \cong H^*(Y)$
- but for non-small H **issue: non-compactness** $\implies$ need careful choice of H + further constraints on Y $\implies$ get **symplectic cohomology** $SH^*(Y)$.

Question

What good can the existence of a pseudo-holomorphic *contracting* $\mathbb{C}^*$-action on (Y, ω) give to a symplectic topologist?

Some possible answers (but there may be more!):

1. In the presence of orthogonal **holomorphic** symplectic structure $\omega_{\mathbb{C}}$ of homogeneity 1 $(t \cdot \omega_{\mathbb{C}} = t\omega_{\mathbb{C}}) \implies$ get *special* exact $\omega_{\mathbb{C}}$-Lagrangian submanifolds [Ž.'22].
2. Can construct symplectic cohomology and, consequentially, induce a filtration on ordinary/quantum cohomology of Y [Ritter - Ž.'23].

Today, I will talk about the latter.

Basic example: $\mathbb{C}^n$, three observations

$(\mathbb{C}^n, \omega_{std} = \sum_i dx_i \wedge dy_i)$, standard $\mathbb{C}^*$-action $t \cdot z = tz$. The S^1-**moment map** (=Hamiltonian) is $H = \frac{1}{2}||z||^2$.

Observation I

- There is a certain **maximum principle** ($\mathbb{C}^n$ is *Liouville*) $\implies$
  - well defined $HF(F)$, for $F = \lambda H$ at infinity
  - well-defined continuation maps $HF(F_1) \to HF(F_2)$, for $\lambda(F_1) < \lambda(F_2)$.
- Define $SH(\mathbb{C}^n) := \lim_{\lambda(F)\to\infty} HF(F)$.

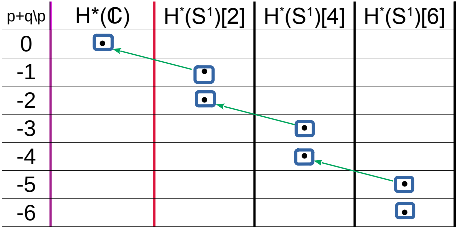
Observation II

- $c_1(\mathbb{C}^n) = 0 + H^1(\mathbb{C}^n) = 0 \Rightarrow$ canonical $\mathbb{Z}$-grading on HF^* and SH^*.
- $SH^*(\mathbb{C}^n) = \lim_{\lambda\to\infty} HF^*(\lambda H) = \lim_{\lambda\to\infty}\langle 0\rangle[2n\lfloor\lambda\rfloor] = 0$.

Basic example: $\mathbb{C}^n$, three observations

Observation III

- "Convex" Hamiltonians $H_\lambda = c(H)$, c convex and $= \lambda H$ at ∞.
- $CF^*(H_\lambda)$ filtered (by $\mathcal{A}_{H_\lambda}$), and the filtration follows the value of H.
- Morse–Bott–Floer **spectral sequence** $E_r^{pq} \Rightarrow SH^{p+q}(\mathbb{C}^n)$,
  $E_1^{0q} = H^q(\mathbb{C}^n)$, $E_1^{pq} = H^q(S^{2n-1})[2pn]$



Questions:

1. Generalise beyond $\mathbb{C}^n$ such that Observations I-III hold?
2. Applications?

Let us start answering Question 1 first.

Symplectic $\mathbb{C}^*$-manifolds

Definition

Symplectic $\mathbb{C}^*$-manifold is a connected symplectic manifold (Y, ω, I) admitting a pseudoholomorphic $\mathbb{C}^*$-action φ whose S^1-part is Hamiltonian.

- Assume $\mathbb{C}^*$-action is *contracting*, $\mathfrak{F} := Y^{\mathbb{C}^*}$ is compact and $\forall y, \exists \lim_{\mathbb{C}^* \ni t \to 0} t \cdot y \in \mathfrak{F}$.
- The other limit defines the $\mathrm{Core}(Y) := \{y \in Y \mid \exists \lim_{\mathbb{C}^* \ni t \to \infty} t \cdot y\}$.

Theorem

1. $\mathrm{Core}(Y)$ is compact and connected.
2. It is deformation retract of Y when $(Y, Core(Y))$ CW-pair
3. $H^*(Y) \cong \check{H}^*(\mathrm{Core}(Y))$ ($\cong H^*(\mathrm{Core}(Y))$ when CW complex)
4. Although the moment map $H : Y \to \mathbb{R}$ might not be proper, $H^*(Y) \cong \bigoplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\alpha]$ (Atiyah–Bott filtration),
5. In particular $\exists! \mathfrak{F}_{\min}$ minimum of H **(minimal component)**.

Symplectic $\mathbb{C}^*$-manifolds over a convex base

- Attempt to define $SH^*(Y)$ as for $\mathbb{C}^n$ **(Observation I)**.
  Issue: Usual max principle a priori does not work.
- Motivated by Conical Symplectic Resolutions, we fix this by imposing further: Assume that there is a proper map

$$\Psi : (Y \setminus \text{compact}, I) \to (\Sigma \times [R_0, \infty), I_B), \;\; \Psi_* X_{S^1} = (f > 0) \cdot \mathcal{R}_B.$$

- Such $(Y, \omega, I, \varphi, \Psi)$ we call **Symplectic $\mathbb{C}^*$-manifolds over a convex base.**
- In particular, when have a global $\Psi : Y \to B$ for a convex B, call Y **globally defined** (almost all examples)
- **Main examples:** Equivariant projective morphisms $p : Y \to X$ to affine X with a contracting $\mathbb{C}^*$-action.
  Here equivariantly embed $X \subset \mathbb{C}^n =: B$ and compose with p to get Ψ.

Examples

1. Equivariant projective morphisms $\Psi : Y \to X$ to affine X with a contracting $\mathbb{C}^*$-action.
   1. $T^*\mathbb{CP}^1 \to \mathbb{C}^2/\mathbb{Z}_2 \cong V(XY - Z^2) \subset \mathbb{C}^3$
   2. Minimal resolution $Y_G \to \mathbb{C}^2/G$, for $G \leq SL(2, \mathbb{C})$ finite
   3. Crepant resolutions $Y \to \mathbb{C}^n/G$, for $G \leq SL(n, \mathbb{C})$ finite.
   4. Conical symplectic resolutions **(CSRs)** $\Psi : (Y, \omega_{\mathbb{C}}) \to X$, with $t \cdot \omega_{\mathbb{C}} = t^{s>0}\omega_{\mathbb{C}}$. (Springer resolutions, quiver var., hypertoric var.,...)
   5. Higgs moduli spaces, $\Psi : \mathcal{M} \to \mathbb{C}^N$ Hitchin fibration.
   6. Kähler quotients $\Psi : Y = \mathbb{C}^n /\!\!/_{\zeta} G \to \mathbb{C}^n /\!\!/_0 G$, $G \leq U(n)$
      In particular, for $G =$ torus $\implies$ Semiprojective toric varieties,
2. Trivial vector bundles $B \times \mathbb{C}^r \to \mathbb{C}^r$, B symplectic
3. Negative vector bundles $E \to B$, with $\Psi : E \setminus 0 \to L \setminus 0$, for $L \to \mathbb{P}(E)$ tautological (only here Ψ is not global)
4. $\mathbb{C}^*$-invariant submanifolds, e.g. fixed locus of $\mathbb{Z}/m \leq \mathbb{C}^*$.
5. Equivariant blow-ups

NB. For 1.(1-5) have $c_1(Y) = 0$, for others not in general.

Back to Observations I and II

Theorem (Construction of SH)

Given a symplectic $\mathbb{C}^*$-manifold over a convex base (Y, ω, I, φ), $SH(Y,\varphi) := \varinjlim_\lambda HF(F)$ is a well-defined unital ring ($F = \lambda H$ at infinity) When $\Psi : Y \to X$ is morphism over affine X and actions φ_1, φ_2 commute, $SH^*(Y,\varphi_1) \cong SH^*(Y,\varphi_2)$.

Considering "clean" Hamiltonians λH, for φ-generic λ, we get:

Proposition

If $c_1(Y) = 0$, $SH^*(Y,\varphi) = \lim_{\lambda\to\infty} HF^*(\lambda H) = 0$.

Conjecture

Let $Y = \mathbb{C}^{2n} /\!\!/\!\!/_\zeta G$, $G \leq Sp(n)$ and $L \subset (Y, \omega_J)$ a closed exact Lagrangian. Then $H^2(L;\mathbb{R}) \neq 0$ and $\pi_2(L)$ is infinite.

Application: Filtration on $QH^*(Y)$

Proposition

$\exists$ Floer-theoretic filtration $\mathscr{F}^{\varphi}_{\lambda}(QH^*(Y))$ by ideals on the ring $QH^*(Y)$. If $SH^*(Y) = 0$, it exhausts it, otherwise define $\mathscr{F}^{\varphi}_{+\infty} := QH^*(Y)$

- canonical $c^*_\mu : QH^*(Y) \cong HF^*(F_{\text{small slope}}) \to HF^*(F_\mu)$
- filtration $\mathscr{F}^{\varphi}_{\lambda} := \bigcap_{\text{generic } \mu > \lambda} (\ker c^*_\mu)$ "survival time"
- $\mathscr{F}^{\varphi}$ is compatible with grading $\implies$ get filtrations $\mathscr{F}^{\varphi}(H^k(Y))$.
- Although $SH^*(Y, \varphi)$ is usually φ-independent, $\mathscr{F}^{\varphi}$ **can** depend on φ!
- Specialise at $T = 0$ (Novikov parameter) $\implies$ get filtration $\mathscr{F}^{\varphi}_{\mathbb{B},\lambda}$ of $H^*(Y, \mathbb{B})$ by cup-ideals, $\mathrm{rk}_{\mathbb{K}} \mathscr{F}_\lambda = \mathrm{rk}_{\mathbb{B}} \mathscr{F}_{\mathbb{B},\lambda}$.

Example: For CSRs get a family of filtrations on $H^*(Y)$, labelled via convex set of contracting $\mathbb{C}^* \leq MaxTor(Y)$

Lower bounds on filtration

- Using clean Hamiltonian λH we get the energy spectral sequence

$$\oplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\lambda(\mathfrak{F}_\alpha)] \implies HF^*(\lambda H)$$

  where $\mu_\lambda(\mathfrak{F}_\alpha)$ computable via **weights** $T_{\mathfrak{F}_\alpha} Y = \oplus \mathbb{C}_{w_i}$.
- When $H^{odd}(Y) = 0$ (e.g. all CSRs), get

$$\oplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\lambda(\mathfrak{F}_\alpha)] \cong HF^*(\lambda H)$$

- The continuation maps $c^*_\lambda : \oplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\alpha] \to \oplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\lambda(\mathfrak{F}_\alpha)]$

$$\mathrm{rk}\,(\mathscr{F}_\lambda(H^k(Y)) \geq \sum_\alpha b_{k-\mu_\alpha}(\mathfrak{F}_\alpha) - b_{k-\mu_\lambda(\mathfrak{F}_\alpha)}(\mathfrak{F}_\alpha).$$

  Example: for weight-s CSR get $\mathscr{F}^\varphi_1 = H^*(Y)$ for $s \geq 2$, and $\mathscr{F}^\varphi_1 \supset H^{\geq 2}(Y)$, $\mathscr{F}^\varphi_2 = H^*(Y)$ for $s = 1$.

Filtration at integer times: The Q_φ invariant

- Following [Ritter'14], $c^*_{N^+} = Q_\varphi^{\star N} \star \cdot : QH^*(Y) \to HF^*(H_{N^+})$, where $Q_\varphi \in QH^{2\mu}(Y)$ is the generalised Seidel element of φ.

  $\implies \mathscr{F}_N^\varphi = \ker(Q_\varphi^{\star N} \star \cdot)$, and
  $c^* : QH^*(Y) \to SH^*(Y, \varphi)$ is the localisation at Q_φ.

Proposition

Suppose Y is Kähler Calabi–Yau/monotone, and $\mathfrak{F}_{\min}$ only has weights 0 and 1. If the Euler class of the normal bundle of $\mathfrak{F}_{\min} \subset Y$ is non-zero,

$$Q_\varphi = PD[\mathfrak{F}_{\min}] + (T^{>0} \text{ terms}) \neq 0,$$

in particular $\mathscr{F}_1^\varphi \neq QH^*(Y)$.

Example: Y is a weight-1 CSR. $\mathscr{F}_1^\varphi = H^{\geq 2}(Y) \subset \mathscr{F}_2^\varphi = QH^*(Y)$

Survival of the minimal component

- Assuming that $H^{odd}(Y) = 0$, recall the continuation map becomes: $c_\lambda^* : \oplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\alpha] \to \oplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\lambda(\mathfrak{F}_\alpha)]$

Proposition

Assume $H^{odd}(Y) = 0$ and $\lambda < 1/(max\ absolute\ weight\ of\ \mathfrak{F}_{\min})$.

$$c_\lambda^*|_{H^*(\mathfrak{F}_{min})} = \mathrm{id}_{H^*(\mathfrak{F}_{\min})[-\mu_\lambda(\mathfrak{F}_{\min})]} + (T^{>0}\text{-}terms),$$

hence

$$\mathscr{F}^{\varphi}_{\mathbb{B},\lambda} \subset \oplus_{\alpha \neq \min} H^*(\mathfrak{F}_\alpha; \mathbb{B})[-\mu_\lambda(\mathfrak{F}_\alpha)].$$

Example: For weight-1 CSR, $\mathfrak{F}_{\min}$ survives until time 1-.

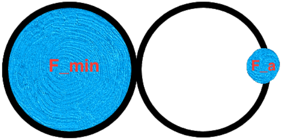
Compare to Atiyah–Bott filtration

- Fixed locus $Y^{\varphi} = \sqcup_{\alpha} \mathfrak{F}_{\alpha}$, U_{α} = upward ∇H-flow of $\mathfrak{F}_{\alpha} \Rightarrow Y = \sqcup_{\alpha} U_{\alpha}$
- Morse breakings induce space filtration
  $\emptyset = W_0 \subset W_1 = U_{\min} \subset \cdots \subset Y$
- Restriction maps $AB_i : H^*(Y) \to H^*(W_i)$
- Filtration on $H^*(Y)$ by cup ideals
  $\ker(AB_i) = \bigoplus_{H(\mathfrak{F}_{\alpha}) \geq H_{i+1}} H^*(\mathfrak{F}_{\alpha})[-\mu_{\alpha}]$
- In particular, $1 \in H^0(\mathfrak{F}_{min})$ lives in the last filtered level, like in $\mathscr{F}^{\varphi}$.
- Unlike A–B, $\mathscr{F}^{\varphi}$ distinguishes different classes in $H^*(\mathfrak{F}_{\alpha})$ in general.

Example: Resolution of A_2 sing. $\mathbb{C}^2/\mathbb{Z}_3 \cong V(XY - Z^3) \subset \mathbb{C}^3$, with action $t \cdot (X, Y, Z) = (tX, t^2 Y, tZ)$.

$\mathscr{F}_{\mathbb{B}} : H^0(\mathfrak{F}_{\alpha})[-2] \subset H^0(\mathfrak{F}_{\alpha})[-2] \oplus H^2(\mathfrak{F}_{\min}) \subset H^0(\mathfrak{F}_{\alpha})[-2] \oplus H^*(\mathfrak{F}_{\min})$,

A–B: $H^0(\mathfrak{F}_{\alpha})[-2] \subset H^0(\mathfrak{F}_{\alpha})[-2] \oplus H^*(\mathfrak{F}_{\min})$.



Observation III: Filtration on Floer chain complex

- Recall: On $\mathbb{C}^n$, used a convex Hamiltonian that is linear at infinity, and the action functional on $\mathbb{C}^n$ (as $\mathbb{C}^n$ is exact).

Theorem

Projecting via $\Psi : Y \to B$, can use the modification of [McLean–Ritter'18] filtration on B to get filtration on $CF^*(H_\lambda)$, that follows the value of moment map H, such that the continuation maps $CF^*(H_\lambda) \subset CF^*(H'_\lambda)$.

Corollary

There is a Morse–Bott–Floer spectral sequence

$$\bigoplus_\alpha H^*(\mathfrak{F}_\alpha)[-\mu_\alpha] \oplus \bigoplus H^*(B_{p,\beta})[-\mu_{p,\beta}] \Rightarrow SH^*(Y,\varphi).$$

where $\sqcup_\beta B_{p,\beta} = \{H = H_p\} \cap Y^{\mathbb{Z}/m}$, $c'(H_p) =: T_p = \frac{2\pi k}{m}$, $(k,m) = 1$.

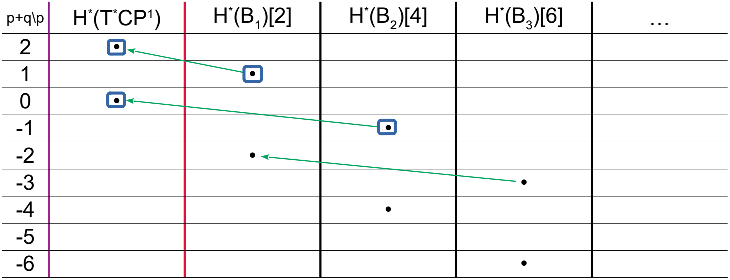
Proposition (Spectral sequence reads the filtration)

$x \in \mathscr{F}^\varphi_\lambda \Leftrightarrow$ *the columns having* $T_p \le \lambda$ *kill* $x \in E_1^{0,q} = H^*(Y)$.

$\implies$ **(Stability)** $\mathscr{F}^\varphi_\lambda = \mathscr{F}^\varphi_{\lambda'}$ *if there are no outer* S^1*-periods* $T_p \in (\lambda, \lambda']$.

Example 1: $Y = T^*\mathbb{CP}^1$

- $Y = T^*\mathbb{CP}^1 \to X = \mathbb{C}^2/\mathbb{Z}_2 \cong V(XY - Z^2) \subset \mathbb{C}^3$ is a blow up.
- $t \cdot (X, Y, Z) = (tX, tY, tZ)$ lifts to $T^*\mathbb{C}P^1$ as fibre-dilation.
- $B_k = \{H = H_p\} \cong S^3/\pm\mathrm{id} = \mathbb{RP}^3$
- The action is free, so no non-integer period columns



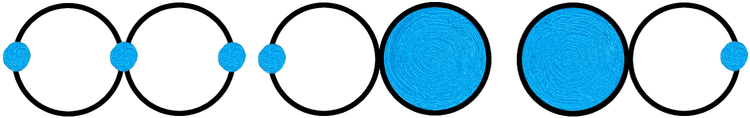
Example 2: $Y = \widetilde{\mathbb{C}^2/\mathbb{Z}_3}$

- Consider $X = \mathbb{C}^2/\mathbb{Z}_3 \cong V(XY - Z^3) \subset \mathbb{C}^3$,
- $Y = \text{Blow-up}_0 V(XY - Z^3)$, exceptional locus $= \mathbb{CP}^1 \vee \mathbb{CP}^1$

Three interesting $\mathbb{C}^*$-actions:

- $(t^3 X, t^3 Y, t^2 Z)$, induced by the standard $\mathbb{C}^* \curvearrowright \mathbb{C}^2$.
- $(t^2 X, tY, tZ)$
- $(tX, t^2 Y, tZ)$

Fixed locus $\mathfrak{F}$:



Remark: coloured spheres are minimal $\omega_{\mathbb{C}}$-Lagrangians from [Ž.'22].

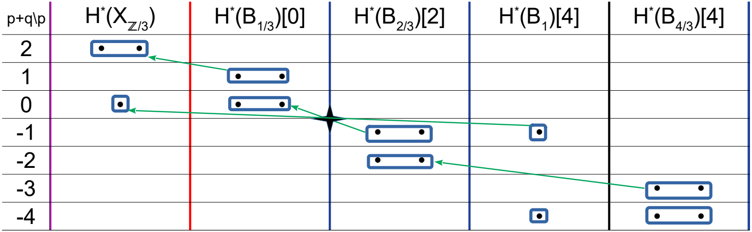
Example 2: $Y = \widetilde{\mathbb{C}^2/\mathbb{Z}_3}$

Morse–Bott manifolds of 1-orbits:

$B_{k/3} \cong (\mathbb{C} \cap \mathrm{slice}) \cong S^1$

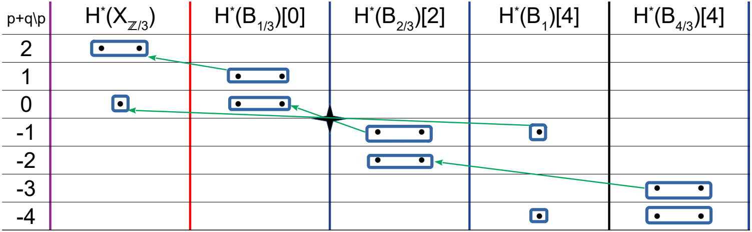
$B_k = (\mathbb{C}^2/\mathbb{Z}_3) \cap \mathrm{slice} = S^3/\mathbb{Z}_3$

Spectral sequence for First action:

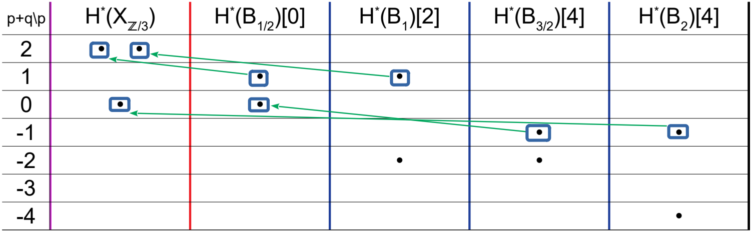


Example 2: $Y = \widetilde{\mathbb{C}^2/\mathbb{Z}_3}$

First action:



Second and third actions:



Note how filtration is different on $H^2(Y)$!

More precisely, we have $\mathscr{F}_{\mathbb{B},\frac{1}{2}} =$ [the non-minimal sphere]

Comparing to other filtrations

- want to compare $\mathscr{F}$ with other filtrations coming e.g. from algebraic geometry or representation theory.
- Compare *degree-wise*, on each $H^k(Y)$.
- For any projective equivariant morphism over affine $\Psi : Y \to X$ weight filtration is trivial (Y semiprojective).
- For certain CSRs there are rep. theoretic filtrations [BS'18].

  For A_n resolutions $\widetilde{\mathbb{C}^2/(\mathbb{Z}/_{n+1})}$, with a particular φ, $\mathscr{F}^\varphi(H^2(Y))$ matches with it rankwise.
- For parabolic Higgs bundles $\Psi : \mathcal{M}_\Gamma \to \mathbb{C}$ of $\dim_{\mathbb{C}} = 2$, filtration $\mathscr{F}_{\mathbb{B}}(H^2(Y))$ is a refinement of the perverse (P=W) filtration of Ψ.

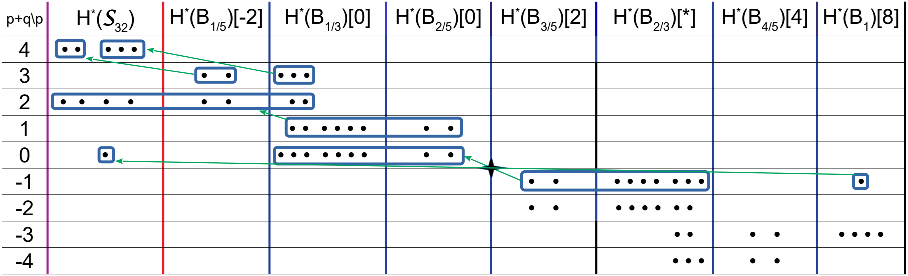
Compare with multiplicity filtration

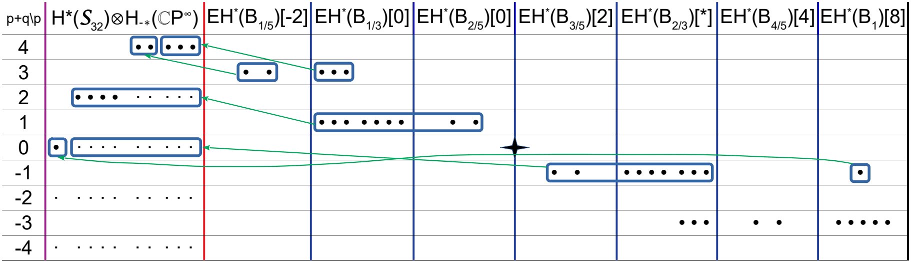
- When $\pi : Y \to X \implies \mathrm{Core}(Y) = \pi^{-1}(0) = \cup_i m_i E_i$ (scheme) when $\mathrm{Core}(Y)$ equidimensional $\implies$ filtration

$$M_k := \{E_i \mid m_i \leq k\} \text{ on } H^{top}(Y).$$

- Question: M_k vs $\mathscr{F}_{\mathbb{B},\lambda}$ on $H^{top}(Y)$?
- $\dim_{\mathbb{C}} = 1$: only $\mathbb{C}$, **equal.**
- $\dim_{\mathbb{C}} = 2$:
  - Higgs examples ($\mathcal{M}_\Gamma$ and 3 others), **equal.**
  - $\Sigma \times \mathbb{C}$, **equal.**
  - CSR examples = ADE resolutions, $X_\Gamma \to \mathbb{C}^2/\Gamma$, $\Gamma \leq SL(2,\mathbb{C})$, $\mathscr{F}$ is a **refinement** of M_k.
- $\dim_{\mathbb{C}} \geq 3$: Unknown (for now...)

Equivariant comparison: Example of $\dim_{\mathbb{C}} = 4$ CSR





The end

Thank you for listening.