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Stated Skein Modules of 3-Manifolds

TQFT Club Lisbon

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Stated Skein Modules of 3-Manifolds
(Joint with T. Le)

Arxiv: 2206.10906

- Basic definitions
- Algebraic properties for surfaces
- The case of 3-manifolds
- A "roots" TQFT

In this talk all surfaces and manifolds will be smooth oriented

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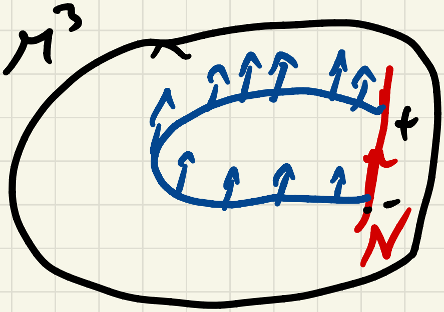
<u>Def</u> A Marked 3-mfld is a pair (M, N) with M an oriented 3-mfld, $N \subset \partial M$ a finite set of embedded oriented arcs or circles (the "marking").

Stated Skein Modules

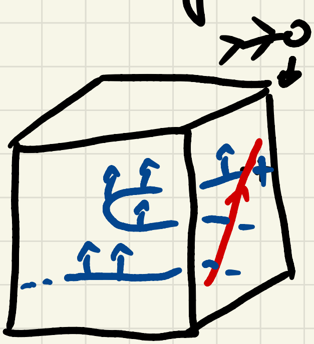
Def A N-link in (M, N) is an isotopy class of framed arcs & circles properly embedded in M, with $\partial \subset N$. The framing is tangent to N along it.

Ex M^3

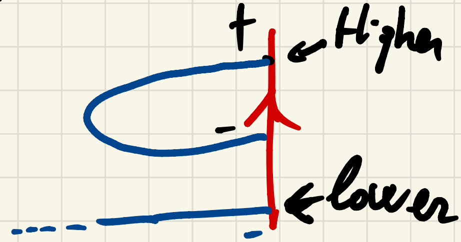


A "state" is a $\pm$ on each ∂ point.

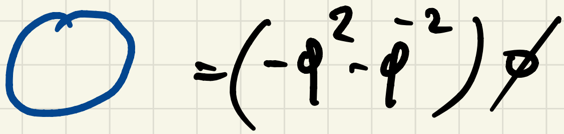
Drawing convention:

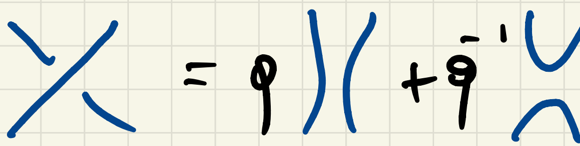


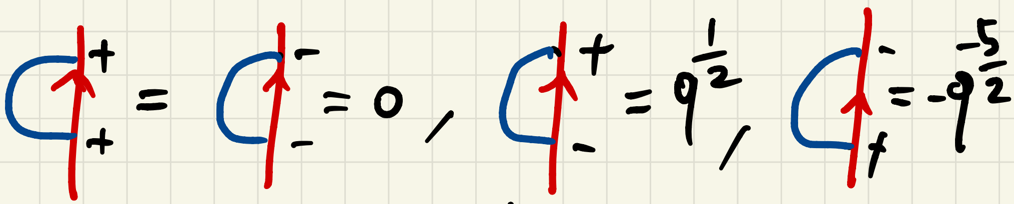
$\rightsquigarrow$

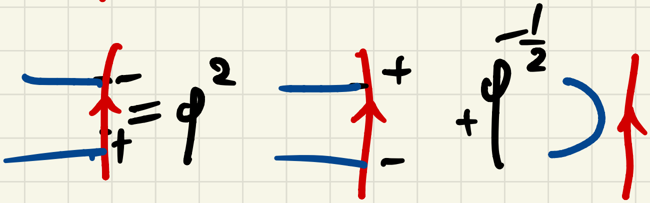


<u>Def</u> The stated skein module of (M, N) is the quotient of the free $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-module generated by isotopy classes of stated N-links by the submodule of the following relations:

K0)


K1)


K2)


K3)


Rem: Actually in the above definition we can replace $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ with any commutative central ring R endowed with a distinguished invertible element "$q^{\frac{1}{2}}$"

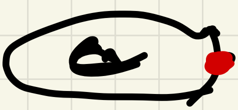
e.g. $R = \mathbb{C}$ & q a root of 1

A special case: marked surfaces –

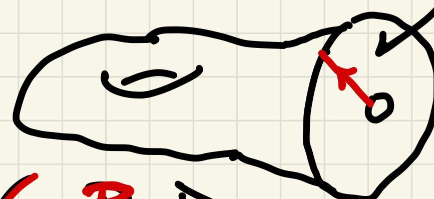
A marked surface is a pair (Σ, P) with $P \subset \partial\Sigma$ a finite set of signed pts.

<u>Idea</u>: To each marked surface one associates a marked 3-mfld "its thickening" via

$\Sigma =$



$(P, +) \rightarrow$ $\quad M = \Sigma \times [-1,1]$ $\quad N = P \times [-1,1]$



The orientation of $N = P \times [-1,1]$ is encoded by $\mathrm{sign}(P)$ –

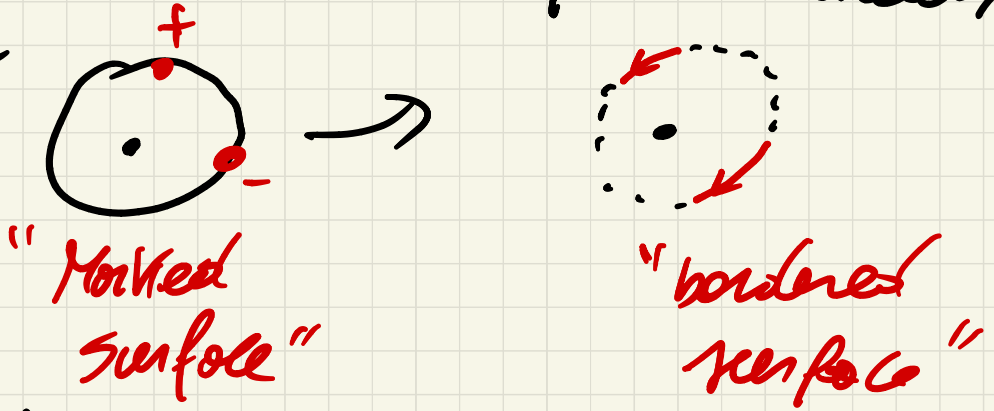
<u>Rem</u> By the previous construction one never gets circle rockings.

Since drawing N-links in marked surfaces is a bit difficult, we adopt a different drawing convention for surfaces.

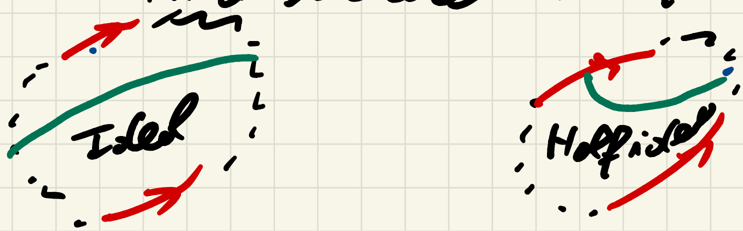
To each marked surface we can associate a "bordered surface" as follows

Idea "bend the red segment, take out the rest of the boundary"

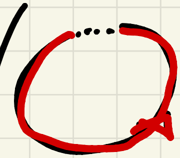
Ex



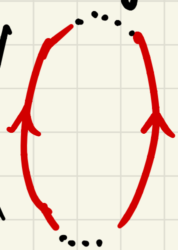
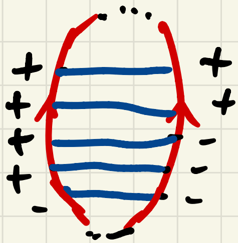
Def An Ideal arc (resp. Half ideal arc) in a bordered surface is:



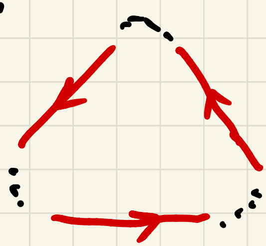
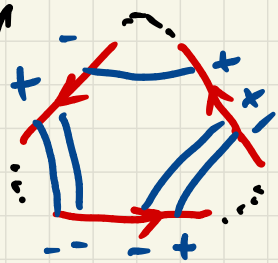
Examples: Marked Surfaces

The Monogon: $\mathcal{S}\Big($  $\Big) = \underbrace{\mathbb{Z}[q^{\pm\frac{1}{2}}]}_{R}$

The Bigon:

$\mathcal{S}\Big($  $\Big) = R\Big\langle\Big\{$  $\uparrow$ increasing states $\Big\}\Big\rangle$

The Triangle:

$\mathcal{S}\Big($  $\Big) = R\Big\{$  $\Big\}$

Some References:

For Surfaces: $\mathcal{S}(\Sigma, P)$ was defined by T. Le, inspired by Bonahon-Wong's construction of the "Quantum trace map" on $\mathcal{S}(\Sigma)$.

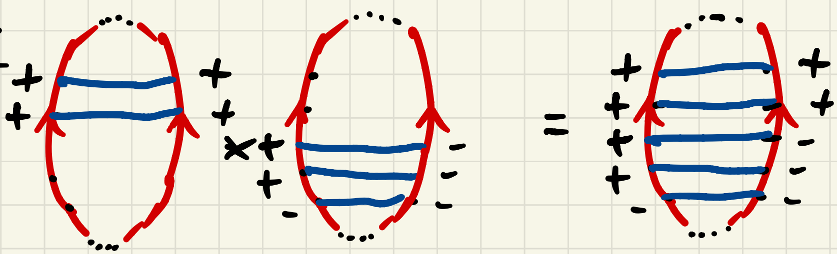
For $P = \{pt\}$ it was proven to be isomorphic to "$\int_{U_q sl_2} \Sigma$" by Cooke.

$\int_{\mathcal{C}} \Sigma$ is the factorization homology with coefficients in a ribbon category $\mathcal{C}$ defined by Ben-Zvi, Brochier & Jordan.

The isomorphism was proven $\forall\, (\Sigma, P)$ by Haïoun.

Thm (T. Le) Let (Σ, P) be a marked surface. Then $\mathcal{S}(\Sigma, P)$ is the free R-module generated by simple P-links with increasing states along $P \times [-1,1]$, and without trivial arcs. Furthermore it is an algebra, with product induced by vertical stacking.

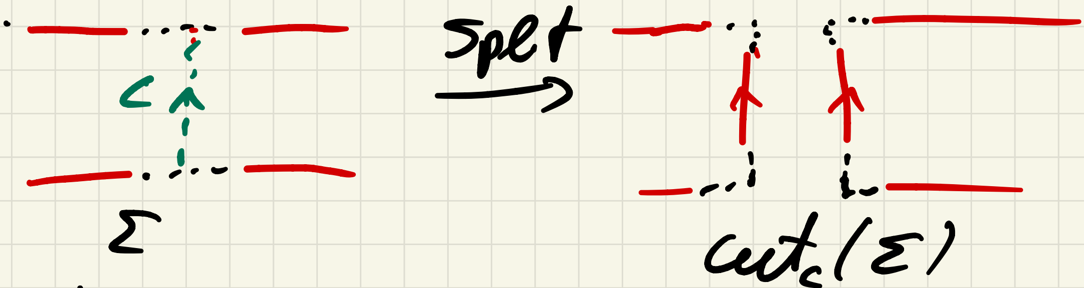
Ex



Where do the relations come from?

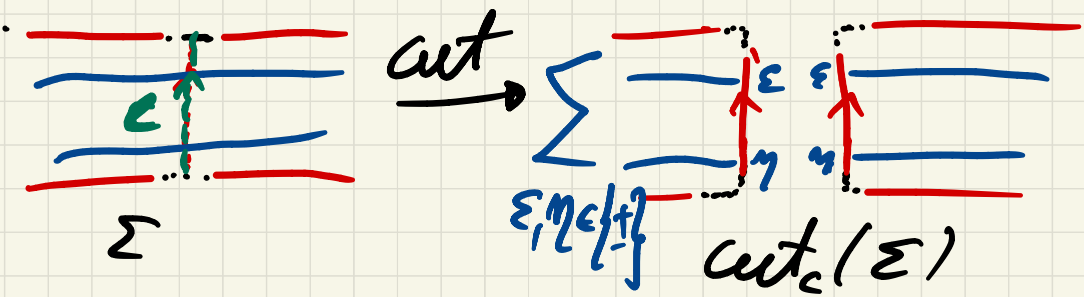
<u>Thm</u> (Splitting, J. Le)

Let (Σ, P) be a marked surface & $c \subset \mathring{\Sigma}$ be an ideal arc:

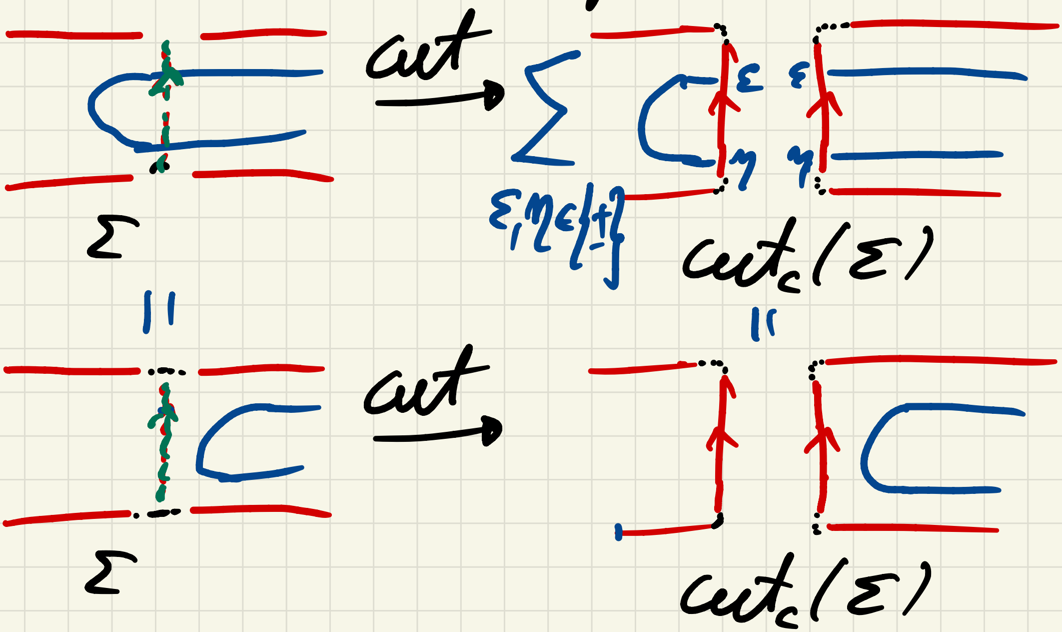


Then the following induces a well defined injective algebra morphism

$$\mathrm{cut}_c : \mathcal{S}(\Sigma) \hookrightarrow \mathcal{S}(\Sigma')$$



Key Point: cutting must be well defined up to isotopy:



Thm (C.–Lê, Korinman–Quesney)

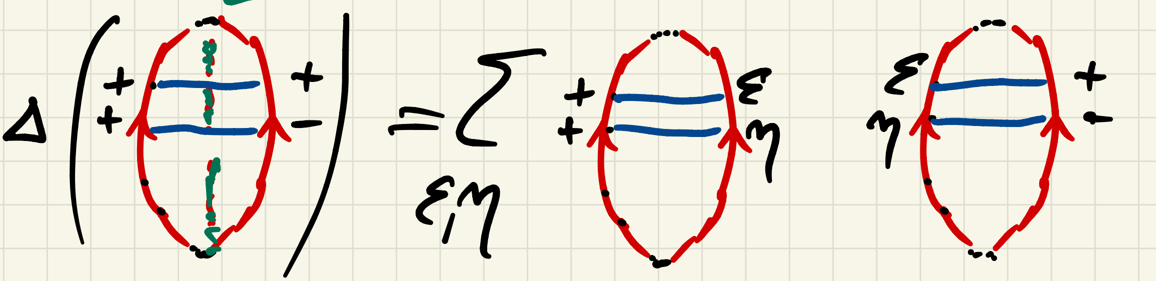
$$\mathcal{S}(\text{Bigon}) \cong \mathcal{O}_{q^2}(SL_2)$$

as Hopf algebras.

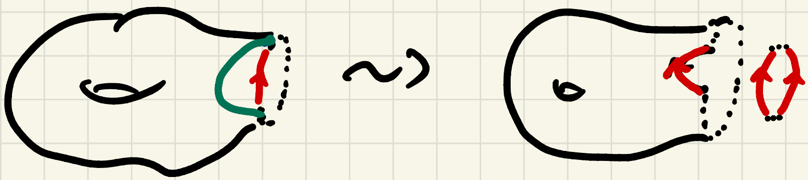
Ex $\Delta : \mathcal{S}(\text{Bigon}) \to \mathcal{S}(\text{Bigon}) \otimes \mathcal{S}(\text{Bigon})$

is:

cut along this arc!

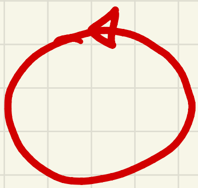


Cor $\forall$ (Σ, P) $\forall$ oriented ∂-arc, cutting out of Σ along a parallel arc yields a $\mathcal{O}_{q^2}(SL_2)$-comodule structure on $\mathcal{S}(\Sigma, P)$:



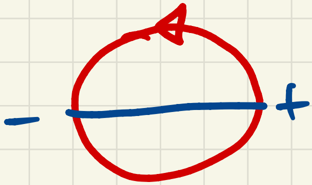
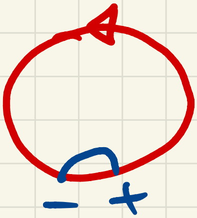
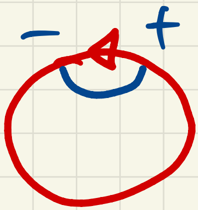
Even better: the structure is compatible with the algebra structure, thus $\mathcal{S}(\Sigma)$ is a $\mathcal{O}_{q^2}(SL_2)$-comodule algebra. $\left(\Delta(xy) = \Delta(x)\Delta(y)\right)$

Recall For general (M, N) we don't have an algebra structure & N might contain circle markings.

Ex $\langle$  $\rangle = \langle$  $\rangle$

$= \dfrac{R}{1+q^2} \rightsquigarrow$ **Not** free!

indeed:

 $=$  $=$ 

$$q^{\frac{1}{2}} \qquad\qquad -q^{\frac{5}{2}}$$

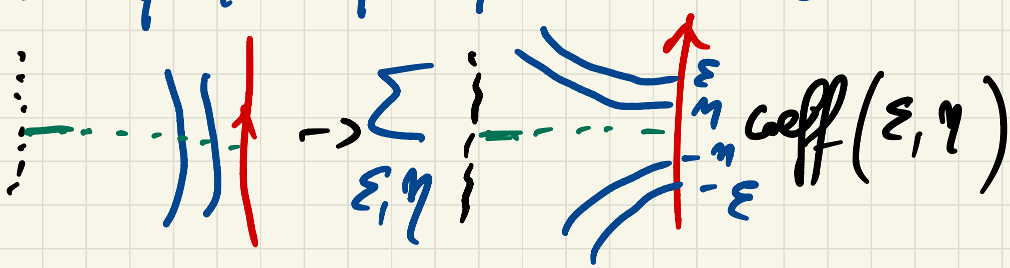
But we could prove the following

<u>Thm</u> (C. - Le)

Let $(\Sigma, \mathcal{P})$ be a connected surface containing at least one ideal arc ($\Rightarrow \Sigma$ non cpt) $\Rightarrow \mathcal{S}(\Sigma, \mathcal{P})$ is <u>free</u> as an R-module.

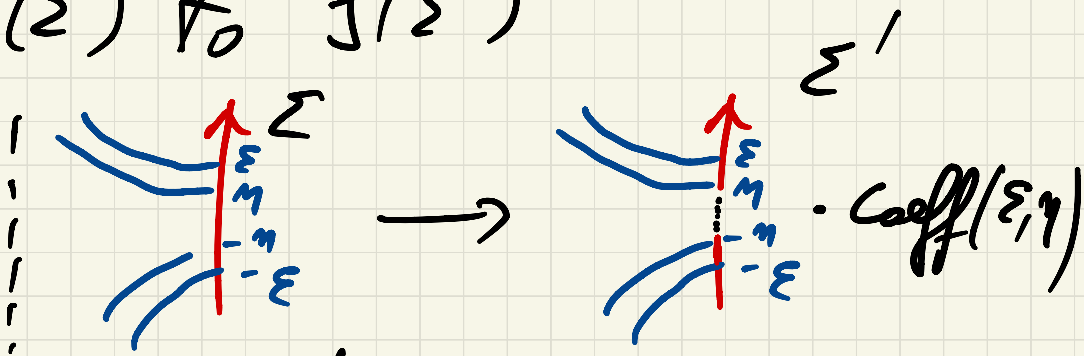
⚠ As soon as there are circle markings $\mathcal{S}(\Sigma, \mathcal{P})$ is <u>not</u> an algebra.

Sketch of Proof of the Theorem:



"splitting operation" along the half ideal arc

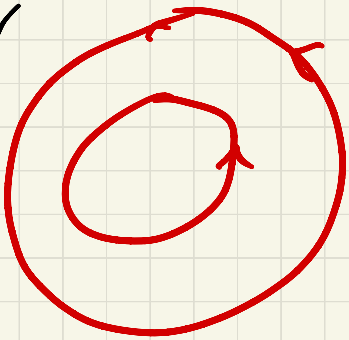
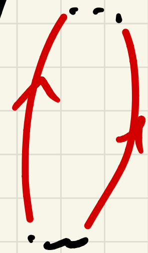
Via slitting one reduces $\mathcal{S}(\Sigma)$ to $\mathcal{S}(\Sigma')$



So that Σ' contains no circle markings $\Rightarrow$ $\mathcal{S}(\Sigma')$ is free as a R-module by the previous results.

Then one proves that slitting which is clearly surjective is also injective.

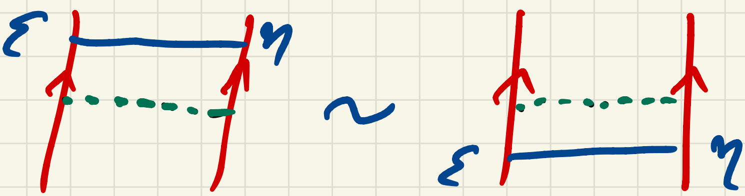
A New Example :

$\mathcal{S}\Big($  $\Big) \cong HH_0\Big($  $\Big)$

More in general :

Thm (C.-L.) If Σ & Σ' are related by slitting a proper arc $\Rightarrow$

$$\mathcal{S}(\Sigma') = \mathcal{S}(\Sigma)/\sim \ :$$



Rem $HH_0\Big($  $\Big)$ is a coalgebra

Ok, but what about 3 Mflds???

Thm (C.-Le) Let (M, N) be a marked 3-mfld and let $c, c' \subset N$ be both oriented arcs or circles. Let $M' = \dfrac{M}{c \sim c'}$ be the 3-manifold obtained by gluing a regular neighborhood of c to one of c'.

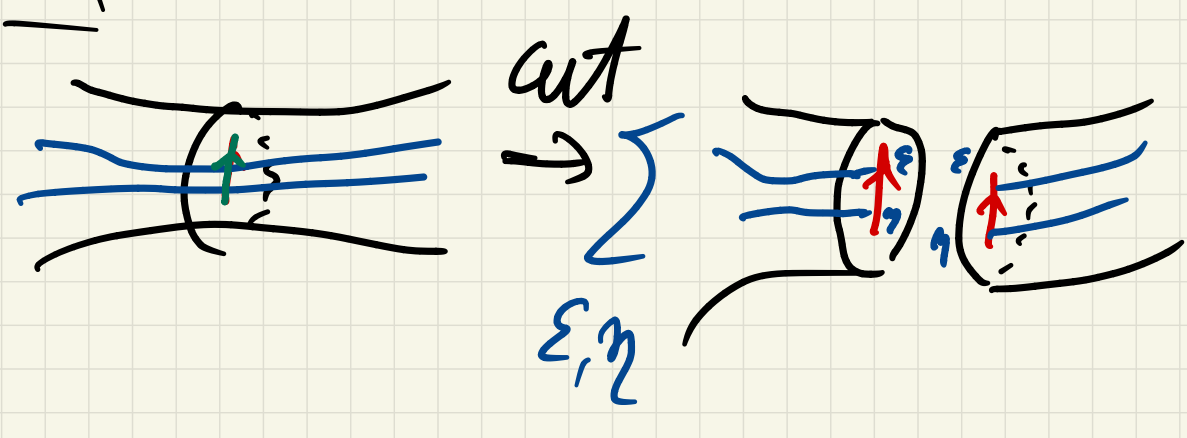
Then cutting (as before) induces a well defined morphism of R-modules:

$$\mathrm{cut} : \mathcal{S}(M', N \setminus \{c, c'\}) \to \mathcal{S}(M, N)$$

⚠ Not necessarily injective!
Not even for c, c' arcs!

<u>Cor</u> $\forall c \in N$ induces a structure of comodule over $\mathcal{O}_{q^2}(SL_2)$ (arc) on $HH_0(\mathcal{O}_{q^2}(SL_2))$ on $\mathcal{S}(M, N)$
(c circle)

<u>Ex</u>

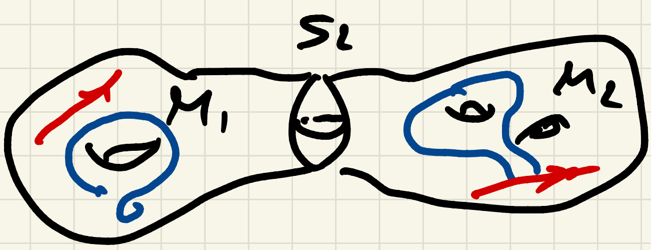


Skein Module of a connected sum.

Let M_1, M_2 be 3-mflds.
Let $R = \mathbb{C}$, $q^{\frac{1}{2}} \neq 0$ be not a $\sqrt[n]{1}$

$\Rightarrow$ As for the standard skein modules one can show that

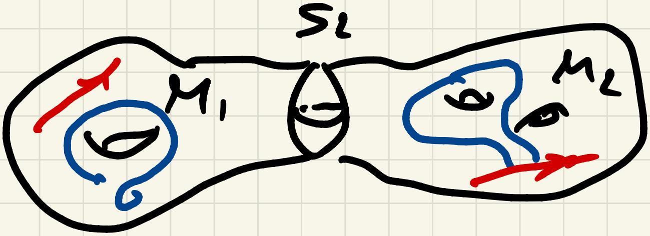
$S(M_1 \# M_2)$ is spanned by skeins which do not intersect the S^2 on which the $\#$ is done:



Surprisingly the situation is quite the opposite if $q^{\frac{1}{2}}$ is a root of 1:

Thm (C.-Le) If both M_1 & M_2 have non empty marking & q is a root of 1 $\Rightarrow$ the skeins disjoint from S^2 are 0 –

↓



$= 0$!

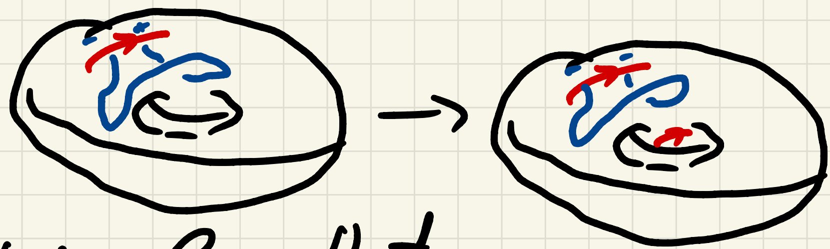
Rem In particular the empty skein $\emptyset = 0$!!!

Forgetting a Marking:

Suppose (M, N) contains a marking arc c and let $N' = N \setminus \{c\}$

Then one has clearly a map induced by inclusion:

$$i_*: \mathcal{S}(M, N') \longrightarrow \mathcal{S}(M, N)$$



It is clear that

$$i_*(\mathcal{S}(M,N')) \subset \mathrm{Coinv}_c(\mathcal{S}(M,N))$$

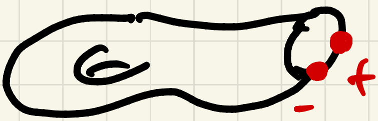
But it turns out that:

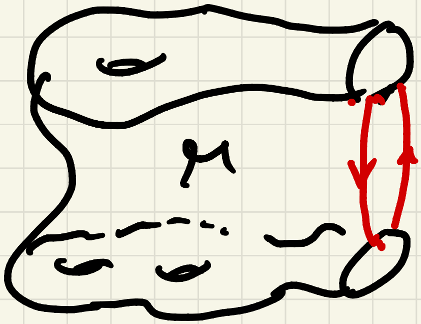
<u>Thm</u> (C.-Lo) If ϕ is a root of 1

$\Rightarrow$ the above map is not injective in general.

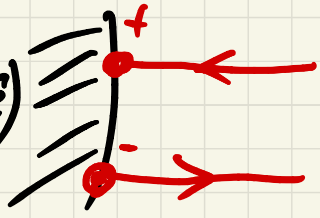
A TQFT FROM STATED SKEINS

Suppose that $i_+ : (\Sigma, P) \hookrightarrow \partial M$ is an orientation preserving embedding as in the drawing:



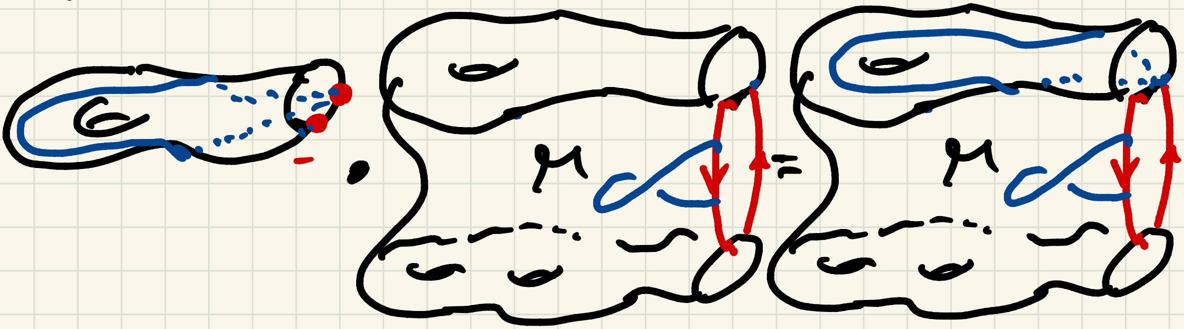


I.e. locally, around $i_+(\partial \Sigma)$:



in ∂M.

Then "Pushing skeins from Σ to M" induces an action of $\mathcal{S}(\Sigma, P) \otimes \mathcal{S}(M, N) \to \mathcal{S}(M, N)$



$\Rightarrow \mathcal{S}(M, N)$ is a left module on $\mathcal{S}(\Sigma, P)$

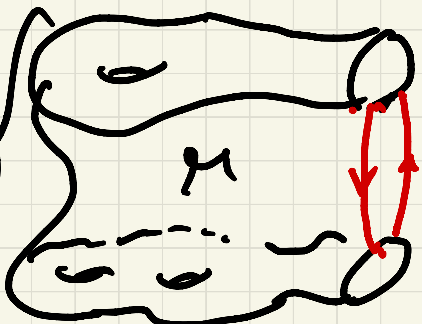
Similarly if $i_-(\Sigma, P) \hookrightarrow \partial M$ reverses the orientation $\Rightarrow$ get a right module structure

<u>Rem</u> (All these actions are compatible with the comodule struct.)

The category of decorated Cob

Let then "dCob" be the category:

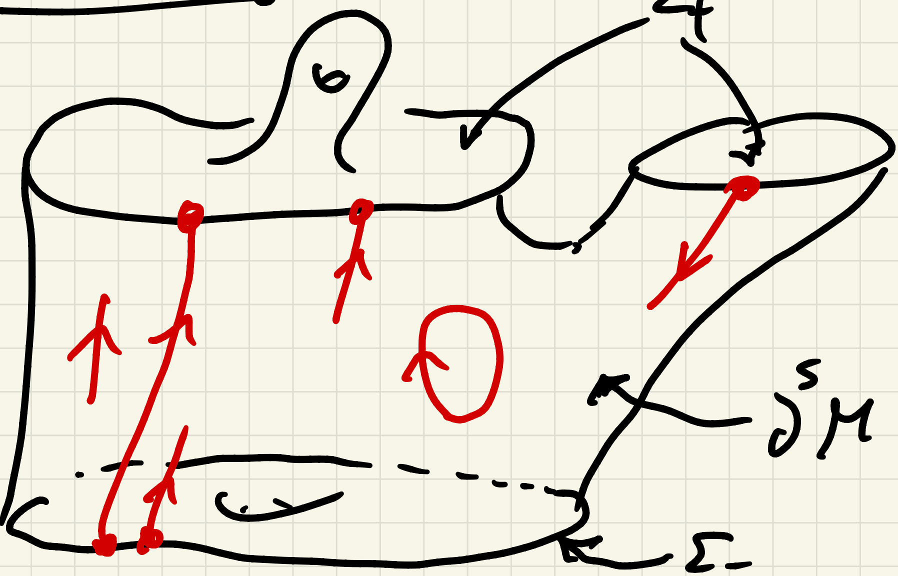
Ob: $\Big\{$  , $P \neq \emptyset \Big\} \sqcup \{\emptyset\}$

Mor: $\Bigg\{$  $: \partial M = \Sigma_+ \sqcup \bar{\Sigma}_- \sqcup \partial^s M$

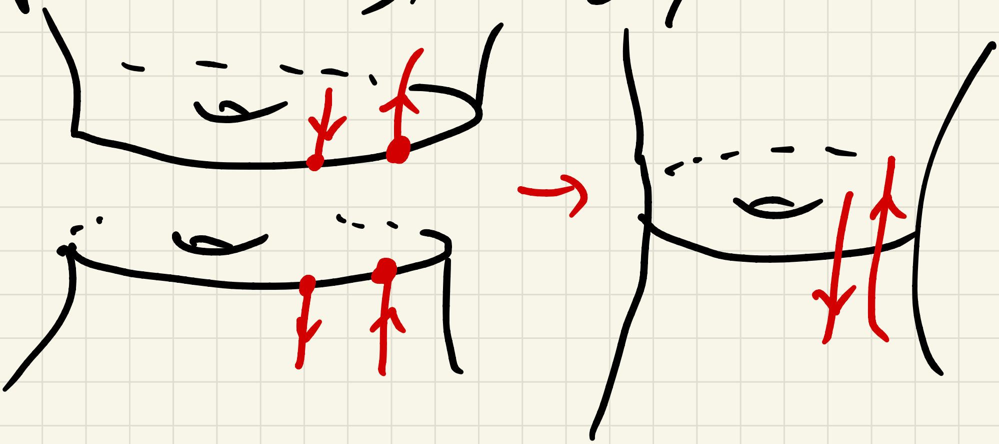
$N \subset \partial^s M \Bigg\}$

(M^3 up to diffeo$^+$)

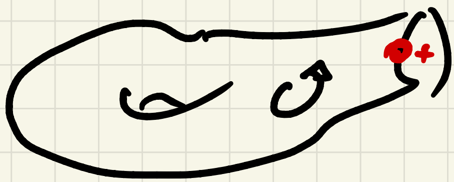
Ex

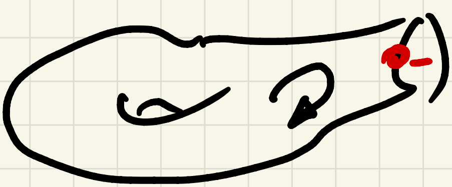


The composition is glueing of cobordisms, locally:



The $\otimes$ is $\sqcup$ $(\emptyset = \mathrm{id})$

The dual of 

is 

Thm (C.-Le) $\mathcal{F}$: Decob $\to$ Morita is a symmetric monoidal functor.

"Morita": objects = Associative unital Algebras

$$\mathrm{Mor}(A_1, A_2) = \frac{\{(A_2, A_1)\text{-Bimods}\}}{\text{iso.}}$$

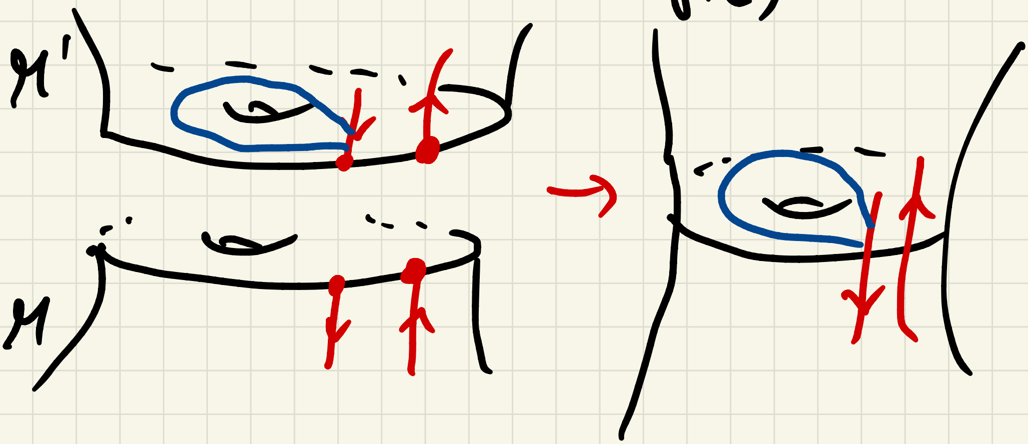
Composition:

$$ {}_{A_3}B'_{A_2} \circ {}_{A_2}B_{A_1} = B' \otimes_{A_2} B = \frac{B' \otimes B}{\{b'a \otimes b - b' \otimes ab\}}$$

What is the Key Point?

To prove that

$$S(M' \circ M) = S(M') \otimes_{S(\Sigma)} S(M)$$



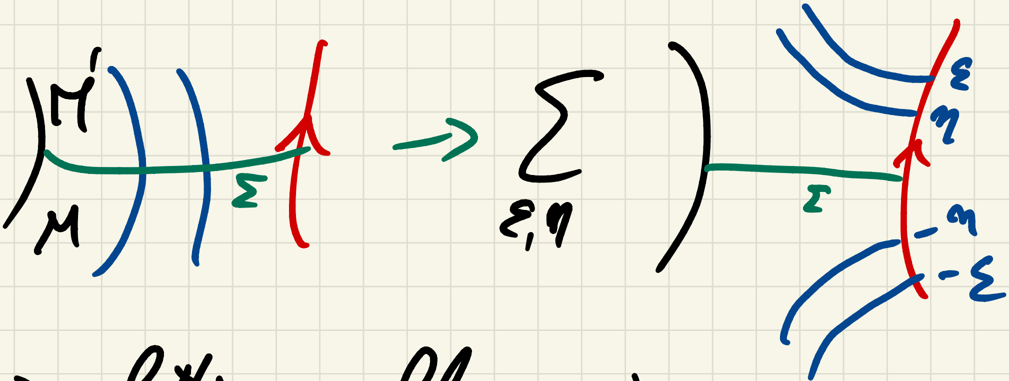
Easy: $\exists$

$$\begin{tikzcd} S(M') \otimes S(M) \arrow[r] \arrow[d, dashed] & S(M' \circ M) \\ S(M') \otimes_{S(\Sigma)} S(M) \arrow[ur, hook] & \end{tikzcd}$$

To prove: $\longrightarrow$ is surjective & injective

The key tool is "slitting" again.

In section:



$\Rightarrow$ slitting allows to prove that any skein in $M' \circ M$ can be split in a sum of skeins in $M' \sqcup M\_$ $\rightsquigarrow$ surjectivity

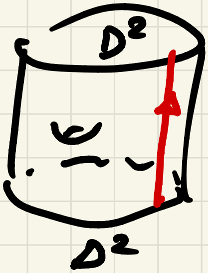
Injectivity: one checks that the relations in $S(M' \circ M)$ can be suitably lifted to $S(M') \otimes_{S(\Sigma)} S(M)$ ...

Some Corollaries:

① $\mathcal{S}(\Sigma\times S^1, P\times S^1) = HH_0(\mathcal{S}(\Sigma, P))$

② $\forall$ closed M^3 let

$\hat{M} = (M^3\setminus \mathring{B}^3, \lambda)$ denoted as:



$\partial^+\hat{M} = (D^2, \bullet +)$

$\partial^-\hat{M} = (D^2, \bullet +)$

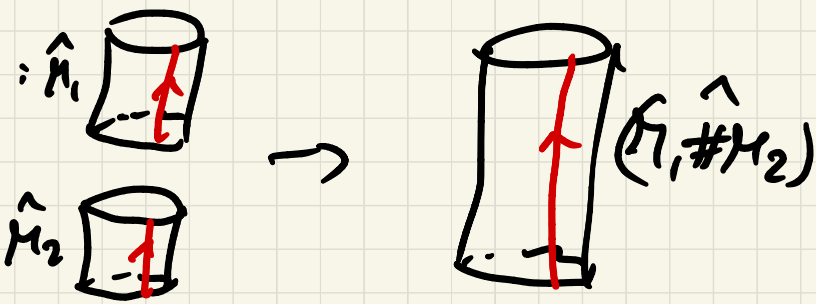
$\partial^s M = S^1\times[0,1]$

$N = \{pt\}\times[0,1]$

Then we have a Van Kampen like theorem:

<u>Thm</u> (C.-Le) $\mathcal{S}(\widehat{M_1\# M_2}) = \mathcal{S}(\hat{M}_1)\otimes_R \mathcal{S}(\hat{M}_2)$

<u>Proof</u>:



$\blacksquare$

Thank You!

Arxiv: 2206.10906