Fano 4-folds with large second Betti number are products of surfaces.

- Smooth, complex Fano variety

Let X be a smooth projective variety.

T_X tangent bundle ($n = \dim X$)

$\det T_X = \bigwedge^n T_X$ = line bundle on X
$n = \dim X$ — anticanonical line bundle $-K_X$

$c_1(-K_X) = c_1(T_X) = c_1(X)$ first Chern class of X

X is FANO if $-K_X$ is ample

equivalently:
a Fano variety is a compact complex manifold with positive c_1

EXAMPLES:

- $\mathbb{P}^n$ is Fano
  $\sim \mathbb{P}^n_{\mathbb{C}}$
- $X \subset \mathbb{P}^{n+1}$ smooth hypersurface of degree $\leq n+1$
- a product of Fano varieties is again Fano

Properties:

- X is simply connected
- X is rationally connected: $\forall x, y \in X$
  $\exists f: \mathbb{P}^1 \to X$ s.t. $f(\mathbb{P}^1) \ni x, y$.

$\exists\ f: \mathbb{P}^1 \to X$ s.t. $\underbrace{f(\mathbb{P}^1)}_{\text{RATIONAL CURVE in } X} \ni x, y$

- <u>boundedness</u>: in every dimension n there are only finitely many families of Fano varieties

i.e.: $\exists\ \mathcal{X}_i \longrightarrow B_i$ $\quad i = 1, .., r$

smooth projective

B_i smooth connected q.p. variety

every fibre is a Fano variety of dim n

and: every n-dim. Fano appears as a fibre in some $\mathcal{X}_i \to B_i$

$\rightsquigarrow$ finitely many diffeomorphism & Topological types.

<u>Classified up to dim 3</u>:

<u>dim $X = 1$</u>: $\mathbb{P}^1$

<u>dim $X = 2$</u>: del Pezzo surfaces

$\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\underbrace{\mathrm{Bl}_{p_1..p_r} \mathbb{P}^2}_{\text{blow up at pts } p_1 \dots, p_r \text{ general},\ r \le 8}$ $\Big]$ $b_2 = 1 + r \le 9$

$\rightsquigarrow$ 10 families.

<u>dim $X = 3$</u>: 105 families

$b_2 = 1$: Iskoskikh & Moscow school '70s

$b_2 > 1$: Mori & Mukai '80s

$\rightsquigarrow$ "Minimal model program"

$b_2(X)$ 2nd Betti number

<u>Thm</u> (Mori–Mukai '86) Let X be a Fano 3fold.

Thm (Mori–Mukai '86) Let X be a Fano 3fold. If $b_2(X)\geq 6$, then $X\cong \mathbb{P}^1\times S$, S a del Pezzo surface.

$$b_2(\mathbb{P}^1\times S) = b_2(\mathbb{P}^1) + b_2(S) = 1 + b_2(S) \leq 10.$$

$\rightsquigarrow$ $b_2 \leq 10$ $\forall$ Fano 3-fold and for $b_2 = 6\dots,10$: only $\mathbb{P}^1\times S$.

Thm (C. '23). Let X be a Fano 4-fold. If $b_2(X)\geq 13$, then $X\cong S_1\times S_2$, S_i del Pezzo surface.

$$b_2(S_1\times S_2) = b_2(S_1) + b_2(S_2) \leq 18$$

$\rightsquigarrow$ $b_2\leq 18$ $\forall$ Fano 4-fold and for $b_2 = 13\dots, 18$: only $S_1\times S_2$.

- all known examples of Fano 4-folds which are not products of surfaces have $b_2\leq 9$

(work in progress: also for $b_2 = 11, 12$ there should be only products of surfaces)

- There are very few examples of Fano 4-folds (not products) with $b_2\geq 6$

  $b_2 = 6$: 10 families (7 toric)

  $b_2 = 7,8,9$: 1 family in each b_2

EXAMPLE (Mukai, C–Codogni–Fanelli 2018).

$\hat{X} = \mathrm{Bl}_{p_1,\dots,p_8 \text{ general}}\mathbb{P}^4$ $\quad b_2(\hat{X}) = 9$ $\quad \hat{X}$ not Fano:

$\downarrow$

$\mathbb{P}^4$

let $\ell\subset\hat{X}$ the transform of a line $\overline{p_ip_j}$

$$\ell\cong\mathbb{P}^1,\quad -K_{\hat{X}}\cdot\ell = -1,\quad \mathcal{N}_{\ell/\hat{X}}\cong\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}$$

Same for: $\ell\subset\hat{X}$ transform of a rational

$(\ldots), \; K_X \cdot \ldots = -1, \; N_{\ell/\hat{X}} = \mathcal{O}_{\mathbb{P}^1}(-1)$

Same for: $\ell \subset \hat{X}$ transform of a rational normal quartic through 7 of the 8 pts

$\rightsquigarrow \binom{8}{2} + 8$ "bad curves", pairwise disjoint

$$\begin{tikzcd} & X' \arrow[dl, "\text{blowup bad curves}"'] \arrow[dr, "\text{blowup of } L's"] & \\ \hat{X} \arrow[rr, dashed, "\text{BIRAT. MAP}"'] & & X \end{tikzcd}$$

smooth proj. FANO, $b_2(X) = 9$

all the exceptional divisors in X' are $E \cong \mathbb{P}^1 \times \mathbb{P}^2$ with $N_{E/X'} \cong \mathcal{O}(-1,-1)$

$E \subset X' \mapsto L \subset X$

$L \cong \mathbb{P}^2$, $N_{L/X'} = \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$

Some recall from birational geometry:

X a Fano 4-fold

A CONTRACTION of X is a surjective morphism $f : X \to Y$ with connected fibres with Y normal, projective

f can be:
- birational if $\dim Y = \dim X$
- of fiber type if $\dim Y < \dim X$

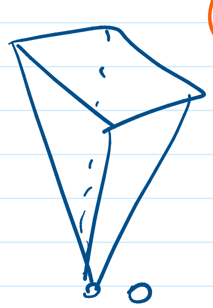
• f is ELEMENTARY if $b_2(X) - b_2(Y) = 1$

$N_1(X) = $ 1-cycles in X, $\mathbb{R}$-coeff. / numerical equivalence — real vector space of dim. $b_2(X)$

$\cup$ $\quad \wr\wr \; H_2(X,\mathbb{R}) \cong H^{2n-2}(X,\mathbb{R})$

$NE(X) = $ convex cone spanned by classes of effective curves (MORI CONE)

$\hookrightarrow$ convex rational polyhedral cone



EXTREMAL RAYS

There are bijections:

There are bijections: $\mathcal{V}_0$ EXTREMAL RAYS ||

$$\left\{\begin{array}{l}\text{elementary}\\ \text{contractions of } X\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{l}\text{1-dimensional}\\ \text{faces of } \overline{NE}(X)\end{array}\right\}$$

$$f : X \to Y \longmapsto \overline{NE}(X) \cap \operatorname{Ker} f_* = \overline{NE}(f)$$

contains classes of curves contracted to pts by f.

$\cap$ $\cap$

$$\{\text{contractions of } X\} \xleftrightarrow{1:1} \{\text{faces of } \overline{NE}(X)\}$$

the face corresponding to f has dim. $b_2(X) - b_2(Y)$

$\overline{NE}(X)$ has dimension $b_2(X)$ $\leadsto$ at least b_2 one dim. faces $\leadsto$ X has at least b_2 elementary contractions

Let $i : D \hookrightarrow X$ be a prime divisor

$$i_* : N_1(D) \longrightarrow N_1(X)$$
$$[C] \longmapsto [C]$$

$N_1(D;X) = i_*(N_1(D)) \subseteq N_1(X)$

$\hookrightarrow$ linear space in $N_1(X)$ of classes of curves in D

$\dim N_1(D,X) \leq b_2(D)$

Let us consider the properties for X:

(a) for every prime divisor, we have

$$\dim N_1(D,X) \geq b_2(X) - 2$$

(b) every elementary contraction of X $f : X \to Y$ is "OF TYPE $(3,2)$", i.e. it is birational with $E := \operatorname{Exc}(f)$ a prime divisor and $S := f(E) \subset Y$ a surface

divisor and $S := f(E) \subset Y$ a surface
(like the blow-up of a surface).

Thm 1 If X does not satisfy property (a) or (b), then: either X is a product of surfaces, or $b_2(X) \leq 12$.

Thm 2 Let X be a Fano 4-fold satisfying (a) & (b). Then $b_2(X) \leq 12$.

Pf of Thm 2: based on a careful study of these contractions of type $(3,2)$

$$\begin{array}{ccc} f: X & \longrightarrow & Y \\ \cup & & \cup \\ E & & S \end{array}$$

simplifying assumption: f is the blow-up of a smooth surface S in a smooth 4-fold Y.

$\Rightarrow$ E is a $\mathbb{P}^1$-bundle over S.

If $\dim \mathcal{N}_1(E, X) \leq 3$, then: by (a) $b_2(X) \leq 5$
ok

$\rightsquigarrow$ we can assume that
$b_2(E) \geq \boxed{\dim \mathcal{N}_1(E, X) \geq 4}$ $\Longrightarrow$ Y is Fano too.

Strategy: to show that $-K_S = (-K_Y)_{|S}$ is ample

$\Rightarrow$ S is a del Pezzo surface

$\Rightarrow$ $b_2(S) \leq 9$ $\Rightarrow$ $b_2(E) = 1 + b_2(S) \leq 10$

$\Rightarrow$ by (a) $b_2(X) \leq 10 + 2 = 12$.

Set $L := (-K_Y)_{|S}$ ample line bundle on S

Consider: $K_S + L$ $\quad (b_2(S) \geq 3)$
nef and defines a contraction
$\varphi: S \to T$ contracting to pts
the curves $C \subset S$ s.t.

$\varphi : S \to T$ contracting to pts

the curves $C \subset S$ s.t.

$(K_S + L) \cdot C = 0$. $\rightsquigarrow$ $K_S + L \equiv 0$.

$\rightsquigarrow$ $-K_S \equiv L = (-K_W)|_S$.