Singular Kähler-Einstein metrics

(joint with Darvas & Lu)

(X^n, ω) compact Kähler manif

complex manif

Kähler form/metric

$(1,1)$-form

closed $(d\omega = 0)$, real $(\omega = \overline{\omega})$

> 0

$$\omega \sim i \sum_{j,k} g_{j\bar{k}} \, dz_j \wedge d\bar{z}_k$$

$(g_{j\bar{k}})$ hermitian metric definite positive

Examples $(n \geq 1)$

- $(\mathbb{CP}^n, \omega_{FS})$

$$\omega_{FS} \sim i\,\partial\bar{\partial}\log(1+|z|^2)$$

Fubini Study

$$\sum \frac{\partial^2}{\partial z_j \bar{z}_k} \log(\quad) dz_j \wedge d\bar{z}_k$$

- $X = \{ [z_0 : \quad : z_{n+1}] \in \mathbb{CP}^{n+1}$ s.t $P(z_0, \dots z_{n+1}) = 0 \}$

hom. poly of degree $d > 0$

$$\omega = \omega_{FS}\big|_X$$

- Complex torus

$$\left(\mathbb{C}^n / \Lambda \ , \ \underbrace{\omega = i \sum_j dz_j \wedge d\bar{z}_j}_{\text{flat metric}} \right)$$

$\hookrightarrow$ lattice

Pb : Given a Kähler manif. X
we look for special metrics
on X.

$\downarrow$

Kähler-Einstein metrics

Def : We say that
$\tilde{\omega}$ is a Kähler-Einstein
metric (KE) if

$\tilde{\omega}$ is Kähler &

(KE) $\boxed{Ric(\underline{\tilde{\omega}}) = \lambda \tilde{\omega}}$ $\quad \lambda \in \mathbb{R}$

$\boxed{\lambda \in \{-1, 0, 1\}}$

Ricci-form $\tilde{\omega}$

$\underbrace{(1,1)\text{-form}}$ closed

$\underset{loc}{\sim} -i\partial\bar{\partial} \log \det(\tilde{g}_{j\bar{k}})$

Obs (KE) is an eq. between (1,1)-forms

$\longleftrightarrow$ a system of equations

⚠ there are topological obstruction to the existence of a KE metric

NECESSARY CONDITION:

$$c_1(X) = c_1(-K_X)$$

> 0	$= 0$	< 0
FANO	CALABI-YAU	GENERAL TYPE
$\mathbb{CP}^n$	$\mathbb{C}^n/\Lambda$	$X \subseteq \mathbb{CP}^{n+1}$ d big enough $(d > n+3)$

Theorem

- $c_1(X) < 0$ then $\exists!\ \tilde{\omega} \in -c_1(X)$ KE, i.e.

$$Ric(\tilde{\omega}) = -\tilde{\omega} \qquad [\lambda = -1]$$

[Aubin-Yau '78]

- $c_1(X) = 0$ Then
$\exists ! \, \tilde{\omega} \in \{\omega\}$ KE i.e
$Ric(\tilde{\omega}) = 0$

[Yau '78]

- $c_1(X) > 0$
there are not KE metrics
$\exists$ KE metric $(Ric(\tilde{\omega}) = \tilde{\omega})$
$\leftrightarrow$ X is K-stable

algebro-geometric condition

[Chen-Donaldson-Sun '14]

Strategy to get existence of KE metrics

(system of eq) $\xleftrightarrow{\text{Kähler}}$ (PDE for a potential)

(KE) $\longleftrightarrow$ complex Monge–Ampère equation

$$(MA)_\lambda \quad \boxed{(\omega + i\partial\bar\partial\varphi)^n = e^{-\lambda\varphi} f\,\omega^n}$$

Kähler form reference

DATA
$0 < f \in C^\infty(X;\mathbb{R})$

solution we are looking for

⚠ $(MA)_\lambda$ is NON-Linear!

Locally

$(MA)_\lambda \longleftrightarrow$

$\lambda = 0$

$$\det\left(g_{j\bar{k}} + \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}\right) = \cancel{e^{-\lambda\varphi}} f \det(g_{j\bar{k}})$$

det is NON-LINEAR !

Theorem

• $\lambda < 0$ [Aubin-Yau '78]

$\exists! \, \varphi \in C^\infty(X; \mathbb{R})$ s.t.

$\omega + i\partial\bar{\partial}\varphi$ is Kähler &

φ is a solution of $(MA)_\lambda$

This is the KE metric !

- $\lambda = 0$ [Yau '78]

$\exists ! \varphi \in C^\infty(X; \mathbb{R})$ solution of $(MA)_0$

$\sup_X \varphi = 0$

- $\lambda > 0$ MORE complicated

TODAY : $\lambda \leq 0$

Goal : Existence of singular Kähler–Einstein metric

Need: define the set of functions I will work with!

$$PSH(X,\omega) = \{\omega\text{-plurisub-harmonic functions}\}$$

Def $u : X \to \mathbb{R} \cup \{-\infty\}$

- u is quasi-psh (qpsh) if $u \underset{\text{loc}}{\sim}$ smooth + psh

- u is ω-PSH if
  u is qpsh &
  $\omega + i\partial\bar{\partial}u \geq 0$ (WEAK

SENSE)

⚠ : Need to compute the second derivative of u in the distributional way

$$i\partial\bar\partial u \geq -\omega$$

Ex: $\boxed{u \sim \log|z|}$ $z = (z_1 .. z_n)$

$$u \sim -(-\log|z|)^{\alpha} \quad \alpha \in (0,1)$$

"Poincaré metric"

$\rightarrow$ $u \sim -\log(-\log|z|)$

$u = |z|^{2\beta}$ $\beta < 1$

FACT

For any u ω-psh function one can define its

Monge-Ampère measure

$(\omega + i\partial\bar\partial u)^n$

~~Ex~~: $\omega + i\partial\bar\partial u$ OK

$(\omega + i\partial\bar\partial u) \wedge (\omega + i\partial\bar\partial u)$ NOT OK

FACT:

- u smooth or bdd

$$\int_X (\omega + i\partial\bar\partial u)^n = \int_X \omega^n$$

[Stokes]

- u singular

$$\underbrace{\int_X (\omega + i\partial\bar\partial u)^n} \le \int_X \omega^n$$

$= m$

Given a mass level

$m \in \,]0, \int_X \omega^n]$

then we want to look at the least singular function with mass equal m.

Given $u \in PSH(X, \omega)$

$$\mathcal{C}(u) = \sup \left\{ v \ \omega\text{-psh} \quad u \le v + C \quad v \le 0 \ \& \ \int_X (\omega + i\partial\bar\partial v)^n = \int_X (\omega + i\partial\bar\partial u)^n \right\}$$

$\mathcal{C} =$ ceiling operator

$e(u)$ called **MODEL POTENTIAL**

$\cap$

$PSH(X,\omega)$

__Ex__: potentials with analytic singularities are model potentials

$$\boxed{u \underset{loc}{\sim} \underset{>0}{c} \underbrace{\log \sum_{j=1}^{N} |f_j|^2 + \mathrm{bdd}}_{\substack{\downarrow \\ \text{holo}}}}$$

$\to \log|f| + \mathrm{bdd}$

$\hookrightarrow$ holo

__theorem__ (DN - Darvas - Lu '20/'21)

Fix Φ a __MODEL POTENTIAL__

(function prescribing the

singularities)

- $c_1(X) \leq 0$ $\exists$ KE metric whose potential φ_{KE} is s.t $|\varphi_{KE} - \phi| \leq C$

[φ_{KE} & Φ has the same type of singularities]

$\boxed{\text{Theorem (DN-Darvas Le)}}$

Fix $\boxed{\phi} \in PSH(X, \omega)$ model potential.

Assume $0 \leq f \in L^p$ $p > 1$ s.t

$$0 < \int_X (\omega + dd^c \phi)^n = \int_X f \omega^n$$

Then

- $\exists!\, u \in PSH(X,\omega)$ solution of

degenerate MA eq. with prescribed singularities

$$\begin{cases} (\omega + i\partial\bar\partial u)^n = f\omega^n & \boxed{\lambda = 0} \\ \sup_X u = 0 \\ |u - \phi| \leq C \end{cases}$$

- $\exists!\, v \in PSH(X,\omega)$ solution of

$$\begin{cases} (\omega + i\partial\bar\partial v)^n = e^{-\lambda v} f\omega^n & \boxed{\lambda < 0} \\ |v - \phi| \leq C \end{cases}$$

$(n=1)$ $\Delta v = e^{-\lambda v} f$

non-pluripolar

RK

- the assumptions on Φ

[mass > 0 & ϕ

model potential]
are needed otherwise
the eq. is NOT well posed

- If $m = \int_X (\omega + i\partial\bar\partial\phi)^n = \int_X \omega^n$

$\rightsquigarrow \Phi = 0$

$\rightsquigarrow$ <u>C^0-estimates</u> for u
OUR THM ($|u| \leq C$)

<u>Dream/Question</u>: Higher estimates?

$\phi \in C^\infty(X \setminus D)$
$\rightsquigarrow u \in C^\infty(X \setminus D)$

$\rightsquigarrow$ I would get a control of $\omega + i\partial\bar\partial u$ in terms of $\omega + i\partial\bar\partial\phi$!

$\boxed{n=1}$

$v = \log|z|$

$\partial\bar\partial \log|z| = \delta_0 \leftrightsquigarrow$ mass $= 0$

non-pluripolar part of $(\ )$ $= 0$

(X, D)

$\longrightarrow$ DN-lm

$\underline{\omega}$ | More control on f

$=$

$\rightsquigarrow$ a better control of u

$$f \leq A e^{-\psi} \qquad \psi = \log \log$$

Fano case $\left(\frac{\text{Xia}}{\text{Tian, Rom}}\right)$

$c_1 > 0$ with ϕ model potential

then ϕ model pot.

$\exists$ KE potential

$|\varphi_{KE} - \phi| \leq C \longleftrightarrow$ Ding functional is proper

a generalised

solving the $(MA)_\lambda$ $\lambda > 0$

VARIATIONAL APPROACH

BBEGZ '13