Embeddings of Symplectic Balls and configuration spaces

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Background

Symplectic Manifolds

Definition

A symplectic manifold M^{2m} is a smooth C^∞-manifold endowed with a 2-form ω which is closed (i.e $d\omega = 0$) and non-degenerate (i.e $\omega^{\wedge n}$ is a volume form).

The non-degeneracy condition is equivalent to saying that the assignment

$$\begin{aligned} TM &\to T^*M \\ X &\mapsto \iota_X\omega \end{aligned}$$

is an isomorphism at every point of M (much like the one defined by a Riemannian metric). The skew-symmetry implies that:

- (M, ω) is of even dimension $d = 2m$.
- (M, ω) is oriented.
- If (M, ω) is closed then $[\omega]^k \neq 0$ in $H^{2k}(M; \mathbb{R})$, for $1 \leq k \leq m$.

Symplectic flexibility

Proposition (Darboux)

Any point p in a symplectic manifold (X, ω) *has a neighborhood symplectomorphic to a standard ball* $B^{2m}(\epsilon) \subset \mathbb{R}^{2m}$ *of small radius* $\epsilon > 0$.

Proposition (Symplectic Neighborhoods)

Let $N_1, N_2 \subset (X, \omega)$ *be two symplectic submanifolds. Suppose there exists a symplectomorphism* $\phi : N_1 \to N_2$ *which lifts to* $\bar{\phi} : T_{N_1}X \to T_{N_2}X$*. Then* $\bar{\phi}$ *can be extended to a neigborhood of* N_1 *and* N_2*.*

Corollary

In dimensions 4*, a neighborhood of a proper, embedded, symplectic surface* $\Sigma_g \hookrightarrow (M^4, \omega)$ *is characterized (symplectically) by g,* $[\Sigma_g] \cdot [\Sigma_g]$*, and* $\omega(S)$*.*

Symplectomorphisms

Definition

A diffeomorphism $\phi : (M, \omega) \to (M, \omega)$ is a symplectic automorphism, or *symplectomorphism*, if $\phi^*\omega = \omega$. We denote by $\mathrm{Symp}(M, \omega)$ the group of all symplectomorphisms of (M, ω).

Given any time-dependent function $H : M \times \mathbb{R} \to \mathbb{R}$, the differential dH corresponds to a unique time-dependent vector field X_H via

$$dH(v) := \omega(X_H, v)$$

Denote by ϕ_t^H the flow of X_H. Then,

$$\mathcal{L}_{X_H}\omega = \iota_{X_H} d\omega + d\iota_{X_H}\omega = 0$$

and it follows that any Hamiltonian flow is symplectic. We denote by $\mathrm{Ham}(M, \omega)$ the subgroup of all Hamiltonian flows. We have inclusions

$$\mathrm{Ham} \subset \mathrm{Symp}_0 \subset \mathrm{Symp}_h \subset \mathrm{Symp} \subset \mathrm{Diff}_{\mathrm{vol}}$$

Proposition (Banyaga)

The group $\mathrm{Ham}(M,\omega) \subset \mathrm{Symp}(X,\omega)$ *is infinite dimensional and acts* N*-transitively on* (M,ω) *for all* $N \geq 1$*. Its isomorphism type completely characterizes* (M,ω)*. If* M *is compact,* Ham *is simple.*

Given a compact symplectic manifold S, let $\mathrm{Emb}_\omega(S, M)$ denote the space of all symplectic embeddings $S \hookrightarrow M$. Let $\Im\mathrm{Emb}_\omega(S, M)$ be the space of submanifolds that are images of such embeddings. There is a fibration

$$\mathrm{Symp}(S) \to \mathrm{Emb}_\omega(S, M) \to \Im\mathrm{Emb}_\omega(S, M)$$

Proposition (Banyaga)

If S *is* 1*-connected, the group* $\mathrm{Ham}(M,\omega)$ *acts transitively on each path component of* $\Im\mathrm{Emb}_\omega(S, M)$*. In other words, each symplectic isotopy* $S_t \subset M$ *can be realized by an ambient Hamiltonian diffeotopy* ϕ_t*, i.e.,* $S_t = \phi_t(S_0)$*.*

Corollary

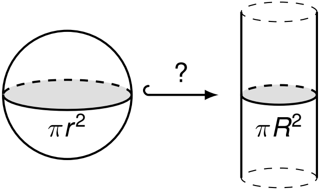
For each path component $\Im\mathrm{Emb}^C_\omega(S, M)$*, there is a fibration*

$$\mathrm{Symp}(M, C) \cap \mathrm{Symp}_0(M) \to \mathrm{Symp}_0(M) \to \Im\mathrm{Emb}^C_\omega(S, M)$$

Nonsqueezing

Theorem (Gromov '85)

If there is an embedding of the closed standard ball of radius r, $B^{2m}(r) \subset \mathbb{R}^{2m}$, in a symplectic product $B^2(R) \times \mathbb{R}^{2m-2}$, then $r < R$.



We define a global symplectic invariant, the *Gromov capacity*, by setting

$$c_{\mathrm{Gr}}(M^{2m}, \omega) = \sup\{\pi r^2 \mid B^{2m}(r) \text{ can be symplectically embedded in } M^{2m}\}$$

Theorem (Eliashberg, Ekeland-Hofer (1989))

If a diffeomorphism $\phi : (M, \omega) \to (M, \omega)$ preserves the Gromov capacity of all symplectic balls, then $\phi^*(\omega) = \pm\omega$.

Spaces of symplectic balls

Consider the space $\mathrm{Emb}(c_1, \dots, c_n; M)$ of symplectic embeddings of the disjoint union of n standard balls of capacities $c_i = \pi r_i^2$ into (M^{2m}, ω), endowed with the C^∞ topology. The space of *unparametrized* embeddings is defined by setting

$$\Im\mathrm{Emb}(c_1, \dots, c_n; M) = \mathrm{Emb}(c_1, \dots, c_n; M) / \operatorname{Symp}(B^{2m}_{c_1}) \times \cdots \times \operatorname{Symp}(B^{2m}_{c_n})$$

Note that Darboux's theorem implies that these spaces are nonempty whenever the capacities $\mathbf{c} := \{c_1, \dots, c_n\}$ are small enough.

The problem

Compute the homotopy type of $\mathrm{Emb}(c_1, \dots, c_n; M)$ or $\Im\mathrm{Emb}(c_1, \dots, c_n; M)$.

This problem encompasses

- the computation of the the Gromov capacity,
- the computation of the packing numbers,
- the symplectic camel problem.

Preliminary observations

Given $\mathbf{c} = (c_1, \ldots, c_n)$ and $\mathbf{c}' = \{c'_1, \ldots, c'_n\}$, we say $\mathbf{c} \le \mathbf{c}'$ iff $c_i \le c'_i$ $\forall i$.
Given $\epsilon = \{\epsilon_1, \ldots, \epsilon_n\}$, $\epsilon_i \ge 0$, we write $\mathbf{c} + \epsilon$ for $\{c_1 + \epsilon_1, \ldots, c_n + \epsilon_n\}$.
For each pair $\mathbf{c}$, $\mathbf{c} + \epsilon$, there is a restriction map

$$i_{\mathbf{c}}^{\mathbf{c}+\epsilon} : \mathrm{Emb}_n(\mathbf{c} + \epsilon, M) \to \mathrm{Emb}_n(\mathbf{c}, M). \tag{1}$$

Let $\mathrm{Sp\,Fr}(n, M)$ be the space of symplectic frames at n ordered points in M. Evaluation of the derivatives at the centers of the n balls defines a fibration

$$\mathrm{Emb}_n^{\mathbf{f}}(\mathbf{c}, M) \to \mathrm{Emb}_n(\mathbf{c}, M) \xrightarrow{j_{\mathbf{c}}} \mathrm{Sp\,Fr}(n, M) \tag{2}$$

where $\mathrm{Emb}_n^{\mathbf{f}}(\mathbf{c}, M)$ consists of embeddings with a fixed framing $\mathbf{f}$.
Since the evaluation maps commute with restrictions, there is a map

$$\varinjlim \mathrm{Emb}_n(\mathbf{c}, M) \xrightarrow{j_\infty} \mathrm{Sp\,Fr}(n, M) \tag{3}$$

where the direct limit is taken with respect to reverse inclusions.

Preliminary observations

For (M^{2m}, ω) compact, and for any $n \geq 1$:

1. It is enough to consider $0 < c_n \leq \cdots \leq c_1$. In particular, we view the balls as labelled, hence, ordered.
2. There exists capacities $\mathbf{c}_0 = (c_1, \ldots, c_n)$ such that the induced map

$$\pi_*(j_{\mathbf{c}}) : \pi_*(\mathrm{Emb}_n(\mathbf{c}, M)) \to \pi_*(\mathrm{Sp\,Fr}(n, M))$$

   is surjective for all $\mathbf{c} \leq \mathbf{c}_0$,
3. The map $j_\infty : \varinjlim \mathrm{Emb}_n(\mathbf{c}, M) \to \mathrm{Sp\,Fr}(n, M)$ is a weak homotopy equivalence.

Remark

In general, it is not known whether there exist capacities $\mathbf{c}_0 = (c_1, \ldots, c_n)$ such that the map $\mathrm{Emb}_n(\mathbf{c}, M) \to \mathrm{Sp\,Fr}(n, M)$ is a weak homotopy equivalence for every $\mathbf{c} \leq \mathbf{c}_0$. We do not even know if $\mathrm{Emb}_n(\mathbf{c}, M)$ is connected for small enough $\mathbf{c}$. See the discussion of stability below.

Preliminary observations

Corollary (Anjos-Kedra-P.)

Let (M^{2m}, ω) be a compact, simply connected, symplectic manifold. Let n be a positive integer satisfying the following conditions:

1. $n \geq 4$ *if* $M = S^2$,
2. $n \geq 3$ *if* $H(M; \mathbb{Q}) \simeq \mathbb{Q}[x]/x^k$,
3. $n \geq 2$ *in all other cases.*

Then there exists capacities $\mathbf{c}_0 = (c_1, \ldots, c_n)$ *such that, for every* $\mathbf{c} \leq \mathbf{c}_0$*, the embedding space* $\mathrm{Emb}_n(\mathbf{c}, M)$ *is rationally hyperbolic and the rational cohomology ring* $H^*(\mathrm{Fix}(M, \mathcal{B}_{\mathbf{c}}); \mathbb{Q})$ *is not finitely generated.*

Idea: Under the above conditions on M and n, the space $\mathrm{Conf}_n(M)$ of ordered configurations is rationally hyperbolic, hence $\mathrm{Sp\,Fr}(n, M)$.

Consequence: It is more interesting to understand how the homotopy type of $\mathrm{Emb}_n(\mathbf{c}, M)$ *changes* as the capacities $\mathbf{c}$ vary.

Symplectic blow-ups

Given a symplectic embedding $B_c^{2m} \hookrightarrow (M^{2m}, \omega)$ of a ball of capacity c, we remove its image and collapse the boundary along the Hopf fibration. The result is a symplectic manifold $(\widetilde{M}_\iota, \tilde{\omega}_\iota)$ diffeomorphic to $M \# \mathbb{CP}^{m-1}$.

- The symplectic blow-up $(\widetilde{M}_\iota, \tilde{\omega}_\iota)$ depends only on ω, the capacity c, and on the isotopy class of the embedding ι.
- $\Sigma \simeq \mathbb{CP}^{n-1}$ represents a nonzero class $E \in H_{2n-2}(\widetilde{M}_p; \mathbb{Z})$.
- Any projective line $\mathbb{CP}^1$ in Σ has symplectic area c.
- $[\tilde{\omega}_\iota] = [\omega] - cE$
- $-K_{\widetilde{M}} = -K_M - E$
- Any symplectomorphisms acting $\mathrm{U}(n)$-linearly near p or Σ is compatible with the operation.

Symplectic blow-ups in dimension 4

- An almost complex structure J on (M^4, ω) is compatible with ω if $\omega(Jx, Jy) = \omega(x, y)$ and $\omega(x, Jx) > 0$ for $x \neq 0$. The space $\mathcal{J}(\omega)$ of all compatible J is always nonempty and contractible.
- A J-holomorphic curve is the image of a J-holomorphic map $(\Sigma, j) \to (M, J)$. For compatible J, such a curve is always symplectic. In real dimension 4, the blow-up operation produces an embedded symplectic (-1) sphere Σ which can be made almost complex for generic $J \in \mathcal{J}(\widetilde{M})$.
- Conversely, given an embedded symplectic (-1) sphere Σ in (M^4, ω) of area $\omega(\Sigma) = c$, it can be contracted to yield a symplectic manifold N with a symplectic ball $B^4(c) \subset N$.

Consequently, J-holomorphic curves techniques can be used to probe the embedding spaces Emb and $\Im\mathrm{Emb}$.

Symplectic blow-up in dimension 4

In dimension 4, the symplectic blow-up establishes an equivalence between

- Existence of an embedding $B_{c_1} \sqcup \cdots \sqcup B_{c_n} \hookrightarrow (M, \omega)$ (the packing problem) and existence of a symplectic form $\widetilde{\omega}$ on $\widetilde{M}$ in the cohomology class $[\omega] - c_1 E_1 - \cdots - c_n E_n$.
- Connectedness of $\Im\mathrm{Emb}(c_1, \ldots, c_n; M)$ (the Camel problem) and uniqueness of symplectic blow-ups of M with capacities $c_1 \ldots c_n$.

We are looking for a correspondance of the form

$$\left\{ \begin{array}{c} \text{Compact families of } n \\ \text{disjoint exceptional curves} \\ \text{of areas } c_1, \ldots, c_n \text{ in } \widetilde{M}_n \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Compact families of } n \\ \text{disjoint symplectic balls} \\ \text{of capacities } c_1, \ldots, c_n \text{ in } M \end{array} \right\}$$

Symplectic balls in rational 4-manifolds

Embeddings in 4-manifolds: general setting

Lemma (Gromov '85)

The symplectomorphism group of a standard ball $B^4(c) \subset \mathbb{R}^4$ retracts onto $\mathrm{U}(2)$.

Lemma (Lalonde-P., 02)

Given $\iota : B^4(c_1) \sqcup \cdots \sqcup B^4(c_n) \hookrightarrow (M^4, \omega)$, there are homotopy equivalences

$$\mathrm{Symp}(\widetilde{M}_\iota, \Sigma) \hookleftarrow \mathrm{Symp}^{\mathrm{U}(2)}(\widetilde{M}_\iota, \Sigma) \xrightarrow{Bl} \mathrm{Symp}^{\mathrm{U}(2)}(M, \mathcal{B}) \hookrightarrow \mathrm{Symp}(M, \mathcal{B})$$

where all maps are inclusions except Bl which is induced by the blow-up.

Lemma (Lalonde-P., 02)

Suppose that for some capacities $c_1, \ldots, c_n$ the space of unparametrized embedding is non-empty and connected. There is a homotopy fibration

$$\mathrm{Symp}(\widetilde{M}_{\mathbf{c}}, \Sigma) \to \mathrm{Symp}(M) \to \Im\mathrm{Emb}(c_1, \ldots, c_n; M)$$

Rational 4-manifolds

Let $\mathcal{C}(M,\omega,N)$ be the set of capacities $\mathbf{c} = (c_1,\dots,c_N)$ for which there exists a symplectic embedding $B^4(c_1) \sqcup \cdots \sqcup B^4(c_n) \hookrightarrow (M,\omega)$.

Theorem

Let M be a symplectic rational 4-manifold.

1. *(LM96, LL95) Any two cohomologous symplectic forms on M are diffeomorphic.*
2. *(LL01) There exists a symplectic embedding $B^4(c_1) \sqcup \cdots \sqcup B^4(c_n) \hookrightarrow (M,\omega)$ if, and only if, the cohomology class*

$$[\widetilde{\omega}_{\mathbf{c}}] := [\omega] - c_1(E_1) - \cdots - c_n(E_n) \in H^2(\widetilde{M}_n,\mathbb{R})$$

   pairs strictly positively with all exceptional classes in $\mathcal{E}(\widetilde{M}_n)$, and if it satisfies the volume condition $\langle [\widetilde{\omega}_{\mathbf{c}}]^2, [\widetilde{M}_n]\rangle > 0$.
3. *(McD98) If M is a symplectic rational 4-manifold, then for each $\mathbf{c} \in \mathcal{C}(M,\omega,n)$, the embedding space $\mathrm{Emb}_n(\mathbf{c},M)$ is path-connected. The same holds for $M = B^4(1)$.*

The relative Moser-Kronheimer fibration

- $\mathrm{Diff}_{[\mathbf{c}]}(\widetilde{M}_{\mathbf{c}}, \Sigma)$ diffeomorphisms that preserves the class $[\widetilde{\omega}_{\mathbf{c}}]$ and that leave Σ invariant.
- $\Omega_{\mathbf{c}}(\Sigma)$ the space of symplectic forms cohomologous to $\widetilde{\omega}_{\mathbf{c}}$ and for which Σ is symplectic.
- By the relative Moser's lemma, and using Part (1) of the previous Theorem, one can show that there is a evaluation fibration

$$\mathrm{Symp}(\widetilde{M_{\mathbf{c}}}, \Sigma) \to \mathrm{Diff}_{[\mathbf{c}]}(\widetilde{M}, \Sigma) \to \Omega_{\mathbf{c}}(\Sigma). \tag{4}$$

The relative McDuff homotopy fibration

- Define the space of pairs

  $$P_{\mathbf{c}}(\Sigma) = \{(\omega', J) \mid \omega' \in \Omega_{\mathbf{c}}(\Sigma),\ J \text{ is compatible with } \omega',\ \Sigma \text{ is } J\text{-holomorphic}\}$$

  and the space of compatible almost-complex structures

  $$\mathcal{A}_{\mathbf{c}}(\Sigma) = \{J \text{ is compatible with some } \omega' \in \Omega_{\mathbf{c}}(\Sigma) \text{ and } \Sigma \text{ is } J\text{-holomorphic}\}.$$

- The projection maps $\Omega_{\mathbf{c}}(\Sigma) \leftarrow P_{\mathbf{c}}(\Sigma) \rightarrow \mathcal{A}_{\mathbf{c}}(\Sigma)$ are homotopy equiv.
- The homotopy fiber of the evaluation map $\mathrm{Diff}_{\mathbf{c}}(\widetilde{M}, \Sigma) \rightarrow \mathcal{A}_{\mathbf{c}}(\Sigma)$ is homotopy equivalent to $\mathrm{Symp}(\widetilde{M_{\mathbf{c}}}, \Sigma)$ and the sequence of maps

  $$\mathrm{Symp}(\widetilde{M_{\mathbf{c}}}, \Sigma) \hookrightarrow \mathrm{Diff}_{[\mathbf{c}]}(\widetilde{M}, \Sigma) \rightarrow \mathcal{A}_{\mathbf{c}}(\Sigma). \qquad (5)$$

  induces a long exact sequence of homotopy groups.
- Consequently, as the capacities **c** vary, the homotopy types of $\mathrm{Symp}(\widetilde{M_{\mathbf{c}}}, \Sigma)$ and of $\Im\mathrm{Emb}_n(\mathbf{c}, M)$ change precisely when the homotopy type of the evaluation map $\mathrm{Diff}_{[\mathbf{c}]}(\widetilde{M}, \Sigma) \rightarrow \mathcal{A}_{\mathbf{c}}(\Sigma)$ changes.

Stability of embedding spaces

Definition

Let (M, ω) be a rational 4-manifold and let $\mathbf{c}_0$, $\mathbf{c}_1$ be two sets of capacities in $\mathcal{C}(M, \omega, n)$. We say that $\mathbf{c}_0$ and $\mathbf{c}_1$ are in the same stability component if there exists a continuous family of capacities $\mathbf{c}_t \subset \mathcal{C}(M, \omega, n)$ interpolating $\mathbf{c}_0$ and $\mathbf{c}_1$ for which the homotopy type of $\mathrm{Emb}_n(\mathbf{c}_t, M)$ is constant.

Corollary

By the McDuff homotopy fibration, the stability components of $\mathcal{C}(M, \omega, n)$ *are determined by the homotopy type of the evaluation map* $\mathrm{Diff}_{[\mathbf{c}]}(\widetilde{M}, \Sigma) \to \mathcal{A}_{\mathbf{c}}(\Sigma)$

Here,

$$\mathrm{Diff}_{[\mathbf{c}]}(\widetilde{M}_{\mathbf{c}}, \Sigma) = \{\text{diffeo. preserving the class } [\widetilde{\omega}_{\mathbf{c}}] \text{ and leaving } \Sigma \text{ invariant}\}$$

$$\mathcal{A}_{\mathbf{c}}(\Sigma) = \{J \text{ is compatible with some } \omega' \in \Omega_{\mathbf{c}}(\Sigma) \text{ and } \Sigma \text{ is } J\text{-holomorphic}\}\,.$$

Comparing $\mathcal{A}_{\mathbf{c}}(\Sigma)$ for different capacities

Problem: For two capacities $\mathbf{c}$, $\mathbf{c}'$, find $\mathcal{A}_{\mathbf{c}}(\Sigma) \to \mathcal{A}_{\mathbf{c}'}(\Sigma)$ inducing

$$\begin{array}{ccccc}
\mathrm{Symp}(\widetilde{M_{\mathbf{c}}}, \Sigma) & \longrightarrow & \mathrm{Diff}_{[\mathbf{c}]}(\widetilde{M}, \Sigma) & \longrightarrow & \mathcal{A}_{\mathbf{c}}(\Sigma) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Symp}(\widetilde{M_{\mathbf{c}'}}, \Sigma) & \longrightarrow & \mathrm{Diff}_{[\mathbf{c}']}(\widetilde{M}, \Sigma) & \longrightarrow & \mathcal{A}_{\mathbf{c}'}(\Sigma)
\end{array}$$

Lemma (Weak $b^+ = 1$ $J-$compatible inflation)

Let M be a symplectic 4-manifold with $b^+ = 1$. Given a compatible pair (J, ω) and a J-holomorphic embedded curve Z, there exists a symplectic form ω' compatible with J such that $[\omega'] = [\omega] + tPD(Z)$, $t \in [0, \lambda)$ where $\lambda = \infty$ if $Z \cdot Z \geq 0$ and $\lambda = \frac{\omega(Z)}{(-Z \cdot Z)}$ if $Z \cdot Z < 0$.

It implies that if $[\widetilde{\omega}_{\mathbf{c}'}]$ can be obtained from $[\widetilde{\omega}_{\mathbf{c}}]$ by inflating along a J-holomorphic curve Z for some $J \in \mathcal{A}_{\mathbf{c}}(\Sigma)$, then $J \in \mathcal{A}_{\mathbf{c}'}(\Sigma)$. In particular, if such a curve Z exists for all $J \in \mathcal{A}_{\mathbf{c}}(\Sigma)$, then there is an inclusion $\mathcal{A}_{\mathbf{c}}(\Sigma) \subset \mathcal{A}_{\mathbf{c}'}(\Sigma)$.

Stability chambers for $n \leq 8$ balls in $\mathbf{CP}^2$

Main Fact: On M_n, $n \leq 8$, the J-compatible inflation is only obstructed by the existence of J-holomorphic curves of self-intersection ≤ -1.

- Given $\mathbf{c} \in \mathcal{C}(N)$, let $\mathcal{S}_{\mathbf{c}}^{\leq -1}(\Sigma) \subset H_2(M_n, \mathbb{Z})$ denote the set of homology classes of embedded $\widetilde{\omega}_{\mathbf{c}}$-symplectic spheres of self-intersection ≤ -1 that intersect non-negatively with the exceptional classes $E_1, \ldots, E_n$.
- (ALLP23) For any $1 \leq n \leq 8$ and any $\mathbf{c} \in \mathcal{C}(n)$ the set $\mathcal{S}_{\mathbf{c}}^{\leq -1}(\Sigma)$ is finite.
- Let $\mathcal{S}_{n}^{\leq -1}(\Sigma)$ be the union of the sets $\mathcal{S}_{\mathbf{c}}^{\leq -1}(\Sigma)$ over all capacities $\mathbf{c} \in \mathcal{C}(n)$.
- To each class $A \in \mathcal{S}_{n}^{\leq -1}(\Sigma)$ correspond a linear functional $H^2(M_n, \mathbb{R}) \to \mathbb{R}$ and an associated map $\ell_A : \mathcal{C}(n) \to \mathbb{R}$ defined by setting $\ell_A(\mathbf{c}) := \langle [\widetilde{\omega}_{\mathbf{c}}], A \rangle$. The wall corresponding to $A \in \mathcal{S}_{n}^{\leq -1}(\Sigma)$ is the set of capacities $\mathbf{c} \in \mathcal{C}(n)$ for which $\ell_A(\mathbf{c}) = 0$.
- (ALLP23) The set of walls is locally finite, that is, given any $\mathbf{c} \in \mathcal{C}(n)$, $0 \leq n \leq 8$, there exists an open neighbourhood $U \subset \mathcal{C}(n)$ that meets at most finitely many walls.

Stability chambers for $n \leq 8$ balls in $\mathbf{CP}^2$

Theorem

(Stability, (ALLP23) Theorem 1.3) For each integer $1 \leq n \leq 8$*, the set* $\mathcal{C}(n)$ *of admissible capacities admits a partition into convex regions, called stability chambers, such that*

1. *each chamber is a convex polyhedron characterized by the signs of the functionals* ℓ_A*,* $A \in \mathcal{S}_n^{\leq -1}(\Sigma)$*.*
2. *If two sets of capacities* $\mathbf{c}$ *and* $\mathbf{c}'$ *belong to the same stability chamber, then we have equality* $\mathcal{A}_{\mathbf{c}}(\Sigma) = \mathcal{A}_{\mathbf{c}'}(\Sigma)$*.*
3. *If two sets of capacities* $\mathbf{c}$ *and* $\mathbf{c}'$ *belong to the same stability chamber, then the embedding spaces* $\mathrm{Emb}_n(\mathbf{c}, \mathbf{CP}^2)$ *and* $\mathrm{Emb}_n(\mathbf{c}', \mathbf{CP}^2)$ *are homotopy equivalent.*

Stability chambers for $n \leq 8$ balls in $\mathbf{CP}^2$

It follows that to describe the stability chambers for embeddings of $n \leq 8$ balls in $\mathbf{CP}^2$, it suffices to describe the sets $\mathcal{S}_n^{\leq -1}(\Sigma)$.

Proposition (Zhang 17)

Let J be a tamed almost complex structure on $M_n = M \# n\overline{\mathbf{CP}}^2$ with $n \leq 8$, and let $C = aL - \sum r_i E_i$ be an irreducible curve with $C \cdot C \leq -1$ and $a > 0$. Then the homology class $[C]$ is one of the following:

1. $L - \sum E_{i_j}$,
2. $2L - \sum E_{i_j}$,
3. $3L - 2E_m - \sum_{i_j \neq m} E_{i_j}$,
4. $4L - 2E_{m_1} - 2E_{m_2} - 2E_{m_3} - \sum_{i_j \neq m_i} E_{i_j}$,
5. $5L - E_{m_1} - E_{m_2} - \sum_{i_j \neq m_i} 2E_{i_j}$,
6. $6L - 3E_m - \sum_{i_j \neq m} 2E_{i_j}$.

Stability chambers for $n \leq 8$ balls in $\mathbf{CP}^2$

Corollary

Given a tamed almost-complex structure J on the symplectic blow-up of $\mathbf{CP}^2$ at n disjoint balls of capacities $c_1, \ldots, c_n$, $0 \leq n \leq 8$, an embedded J-holomorphic sphere of self-intersection $C \cdot C \leq -2$ that intersects each of the exceptional classes $E_1, \ldots, E_n$ non-negatively must represent one of the classes listed in the previous Proposition.

Proof.

Let $[C] = aL - \sum r_i E_i$. Then $0 \leq E_i \cdot C = r_i$. Since any J-holomorphic representative of $[C]$ must have positive symplectic area, and since $\langle [\widetilde{\omega}_{\mathbf{c}}], [C] \rangle = a - \sum c_i r_i$, the coefficient a must be strictly positive. $\square$

Theorem (Stability chambers)

For each integer $1 \leq n \leq 8$, the stability chambers of the set $\mathcal{C}(n)$ of admissible capacities are the convex polygonal regions defined by the linear functionals ℓ_A where A is one of the homology classes of self-intersection $A \cdot A \leq -2$ listed in the previous Proposition.

Stability chambers for $n = 1$ or $n = 2$ balls in $\mathbf{CP}^2$

- For $n = 1$, the space of admissible capacities is the interval $(0, 1)$.
- For $n = 2$, it is the polygon $0 < c_2 \leq c_1 < c_1 + c_2 < 1$.
- Since none of the classes in Proposition (Zhang) have self-intersection less than or equal to -2 when $n \leq 2$, the entire space of admissible capacities is itself a stability chamber.
- It follows that the homotopy type of embedding spaces is independent of the choice of capacities. Consequently, for $n \in \{1, 2\}$, and for any admissible capacity $\mathbf{c}$, it follows that

  $$\Im\mathrm{Emb}_n(\mathbf{c}, \mathbf{CP}^2) \simeq \varinjlim \Im\mathrm{Emb}_n(\mathbf{c}, \mathbf{CP}^2) \simeq \mathrm{Conf}_n(\mathbf{CP}^2).$$

  This recovers older results (P08i).
- Note that from the complex point of view, for $n = 1, 2$, every set in $\mathrm{Conf}_n(\mathbf{CP}^2)$ is in general position.

Stability chambers for $n = 3$ balls in $\mathbf{CP}^2$

- The space of admissible capacities consists of triples $\mathbf{c} = (c_1, c_2, c_3)$ satisfying $0 < c_3 \leq c_2 \leq c_1 < c_1 + c_2 \leq 1$.
- The -2 class $A = L - E_1 - E_2 - E_3$ is the only negative homology class of self-intersection $A \cdot A \leq -2$ contained in the list of Proposition (Zhang). The linear functional ℓ_A separates the space of admissible capacities into two chambers, namely $c_1 + c_2 + c_3 < 1$ and $c_1 + c_2 + c_3 \geq 1$.
- Note that from the complex point of view, for a configuration in $\mathrm{Conf}_3(\mathbf{CP}^2)$, either the 3 points are in general position (belonging to 3 distinct lines) or they are aligned.
- When $c_1 + c_2 + c_3 < 1$, decreasing the capacities keeps the triple $\mathbf{c}$ in the same chamber, so that

$$\Im\mathrm{Emb}_3(\mathbf{c}, \mathbf{CP}^2) \simeq \varinjlim \Im\mathrm{Emb}_3(\mathbf{c}, \mathbf{CP}^2) \simeq \mathrm{Conf}_3(\mathbf{CP}^2).$$

Stability chambers for $n = 3$ balls in $\mathbf{CP}^2$

- When $c_1 + c_2 + c_3 \geq 1$, the space $\Im\mathrm{Emb}_3(\mathbf{c}, \mathbf{CP}^2)$ is homotopy equivalent to the space of monotone symplectic embeddings, that is, to $\Im\mathrm{Emb}_3(\mathbf{c}_m, \mathbf{CP}^2)$ with $\mathbf{c}_m = (1/3, 1/3, 1/3)$.
- For the monotone symplectic blow-up $(\widetilde{M}_{\mathbf{c}_m}, \widetilde{\omega}_{\mathbf{c}_m})$, each exceptional classes $E \in \mathcal{E}_3 = \{E_1, E_2, E_3, L - E_1 - E_2, L - E_2 - E_3, L - E_1 - E_3\}$ has symplectic area $1/3$.
- It follows that E is symplectically indecomposable, that is, for every tamed $J \in \mathcal{J}(\widetilde{\omega}_{\mathbf{c}_m})$, E has a unique embedded J-holomorphic representative. Consequently, the space $\mathcal{J}_{\mathbf{c}_m}(\Sigma)$ is the full space $\mathcal{J}(\widetilde{\omega}_{\mathbf{c}_m})$, which is contractible.
- From the fibration

$$\mathrm{Symp}_h(\widetilde{M}_{\mathbf{c}_m}, \Sigma) \to \mathrm{Symp}_h(\widetilde{M}_{\mathbf{c}_m}) \to \mathcal{C}_{\mathbf{c}_m}(\Sigma) \simeq \mathcal{J}_{\mathbf{c}_m}(\Sigma) \simeq *. \tag{6}$$

  we conclude that the stabilizer $\mathrm{Symp}_h(\widetilde{M}_{\mathbf{c}_m}, \Sigma)$ is homotopy equivalent to the full symplectomorphism group $\mathrm{Symp}_h(\widetilde{M}_{\mathbf{c}_m})$.
- By (Evans 13), the latter group is homotopy equivalent to the standard torus $\mathbf{T}^2$ given by the monotone toric action on $\widetilde{M}_{\mathbf{c}_m}$.

Stability chambers for $n = 3$ balls in $\mathbf{CP}^2$

- Since this action is the monotone toric blow-up of the standard toric action on $\mathbf{CP}^2$ that fixes 3 points, we obtain a ladder of homotopy fibrations

$$\begin{tikzcd} \mathrm{Symp}_h(\widetilde{M}_{\mathbf{c}_m}, \Sigma) \arrow[r] & \mathrm{Symp}(\mathbf{CP}^2) \arrow[r] & \Im\mathrm{Emb}_3(\mathbf{c}_m, \mathbf{CP}^2) \\ \mathbf{T}^2 \arrow[r] \arrow[u, hook] & PU(3) \arrow[r] \arrow[u, hook] & PU(3)/\mathbf{T}^2 \simeq U(3)/\mathbf{T}^3 \arrow[u] \end{tikzcd} \tag{7}$$

  in which the first two vertical arrows are homotopy equivalences.
- We conclude that

$$\Im\mathrm{Emb}_3(\mathbf{c}, \mathbf{CP}^2) \simeq PU(3)/\mathbf{T}^2 \simeq U(3)/\mathbf{T}^3$$

  for all choices of capacities $\mathbf{c}$ such that $c_1 + c_2 + c_3 \geq 1$.

Stability chambers for $n = 4$ balls in $\mathbf{CP}^2$

- For $n = 4$, the chambers are defined by the 5 classes $L - E_i - E_j - E_k$ and $L - E_1 - E_2 - E_3 - E_4$.
- Because of the normalization $0 < c_4 \leq \cdots \leq c_1 < 1$, the symplectic areas of these classes are linearly ordered:

$$\begin{aligned} 1 - c_1 - c_2 - c_3 - c_4 < 1 - c_1 - c_2 - c_3 \leq 1 - c_1 - c_2 - c_4 \\ \leq 1 - c_1 - c_3 - c_4 \leq 1 - c_2 - c_3 - c_4 \end{aligned}$$

- Consequently, there are exactly 6 stability chambers that we label and order accordingly:

$$C_5 \prec C_4 \prec C_3 \prec C_2 \prec C_1 \prec C_0$$

  We will write $\mathbf{c} \prec \mathbf{c}'$ whenever $\mathbf{c}$ belongs to a chamber that preceeds the chamber of $\mathbf{c}'$.
- Note that from the complex point of view, the 6 chambers correspond to the relative positions of 4 distinct and ordered points in $\mathbf{CP}^2$.

Stability chambers for $n = 4$ balls in $\mathbf{CP}^2$

We can identify the homotopy type of $\Im\mathrm{Emb}_4(\mathbf{c}, \mathbf{CP}^2)$ for the two extreme chambers.

- When $\mathbf{c} \in C_5$, decreasing the capacities keeps the 4-tuple $\mathbf{c}$ in the same chamber, so that

$$\Im\mathrm{Emb}_4(\mathbf{c}, \mathbf{CP}^2) \simeq \varinjlim \Im\mathrm{Emb}_4(\mathbf{c}, \mathbf{CP}^2) \simeq \mathrm{Conf}_4(\mathbf{CP}^2).$$

- When $\mathbf{c} \in C_0$, then $\Im\mathrm{Emb}_4(\mathbf{c}, \mathbf{CP}^2)$ is homotopy equivalent to the space of monotone embeddings $\mathbf{c}_m = \{1/3, 1/3, 1/3, 1/3\}$. The space of configurations $\mathcal{C}_{\mathbf{c}_m}(\Sigma)$ is contractible, so that

$$\mathrm{Symp}_h(\widetilde{M}_{\mathbf{c}_m}, \Sigma) \simeq \mathrm{Symp}_h(\widetilde{M}_{\mathbf{c}_m}).$$

- By (Evans 13), $\mathrm{Symp}_h(\widetilde{M}_{\mathbf{c}_m})$ is contractible.
- We obtain a fibration with contractible fibers

$$* \simeq \mathrm{Symp}_h(\widetilde{M}_{\mathbf{c}_m}, \Sigma) \to \mathrm{PU}(3) \simeq \mathrm{Symp}(\mathbf{CP}^2) \to \Im\mathrm{Emb}_4(\mathbf{c}_m, \mathbf{CP}^2) \tag{8}$$

  from which we conclude that

$$\Im\mathrm{Emb}_4(\mathbf{c}, \mathbf{CP}^2) \simeq \mathrm{PU}(3)$$

  for all choices of capacities $\mathbf{c}$ such that $c_2 + c_3 + c_4 \geq 1$.

Stability chambers for $n = 5$ balls in $\mathbf{CP}^2$

- The space of admissible capacities consists of 5-tuples $\mathbf{c} = (c_1, \ldots, c_5)$ satisfying $0 < c_5 \leq \cdots \leq c_1 < c_1 + c_2 \leq 1$ and $\sum c_i < 2$.
- The chambers are defined by the 16 classes $L - E_{i_1} - E_{i_2} - E_{i_3}$, $L - E_{i_1} - E_{i_2} - E_{i_3} - E_{i_4}$, and $L - E_1 - E_2 - E_3 - E_4 - E_5$.
- This time, the symplectic areas of these classes form a partial order, so that the cell decomposition of the space of admissible capacities is described by a graph with a more complicated structure.

Homotopy decompositions of Embedding spaces and of Moduli Spaces

Two simple observations

• The moduli space of n distinct, ordered, points in $\mathbf{CP}^2$ is defined as

$$\mathcal{M}_n(\mathbf{CP}^2) := \mathrm{Conf}_n(\mathbf{CP}^2)/\,\mathrm{PGL}(3,\mathbb{C})$$

From a homotopy theoretic point of view, it is better to look at the homotopy orbit (i.e. Borel construction)

$$\mathrm{Conf}_n(\mathbf{CP}^2)_{h\,\mathrm{PGL}(3,\mathbb{C})} := \Big(E\,\mathrm{PGL}(3,\mathbb{C}) \times \mathrm{Conf}_n(\mathbf{CP}^2)\Big)/\,\mathrm{PGL}(3,\mathbb{C})$$

• By (Gromov 85), $\mathrm{Symp}(\mathbf{CP}^2) \simeq \mathrm{PU}(3) \simeq \mathrm{PGL}(3,\mathbb{C})$, so that the homotopy fiber of of the evaluation map

$$\mathrm{ev} : \mathrm{PGL}(3,\mathbb{C}) \to \mathrm{Conf}_n(\mathbf{CP}^2)$$

is given by

$$\begin{tikzcd}
\mathrm{Symp}(\mathbf{CP}^2,\mathbf{p}) \arrow[r] & \mathrm{Symp}(\mathbf{CP}(2)) \arrow[r] & \mathrm{Conf}_n(\mathbf{CP}^2) \\
F_{\mathrm{ev}} \arrow[r] \arrow[u] & \mathrm{PGL}(3,\mathbb{C}) \arrow[r] \arrow[u] & \mathrm{Conf}_n(\mathbf{CP}^2) \arrow[u]
\end{tikzcd}$$

Homotopy decompositions and symplectic stabilizers

- The space $\mathcal{A}_{\mathbf{c}}(\Sigma)$ admits a stratification whose strata $\mathcal{A}_I$ are characterized by the existence of certain embedded J-holomorphic spheres of negative self-intersection.
- For $N = 3$, when $\mathbf{c}$ belongs to the chamber for which $c_1 + c_2 + c_3 \geq 1$, we have $\mathcal{A}_{\mathbf{c}}(\Sigma) = \mathcal{A}_0$, while $\mathcal{A}_{\mathbf{c}}(\Sigma) = \mathcal{A}_0 \sqcup \mathcal{A}_{123}$ when $c_1 + c_2 + c_3 < 1$.
- For $n = 4$, the space $\mathcal{A}_{\mathbf{c}}(\Sigma)$ admits an analogous stratification whose strata are determined by the chamber C_i to which $\mathbf{c}$ belong. More precisely, if $\mathbf{c}$ belongs to the chamber C_i, then $\mathcal{A}_{\mathbf{c}}(\Sigma) = \mathcal{A}_i$ where

$$\mathcal{A}_0 \subset \mathcal{A}_1 := \mathcal{A}_0 \sqcup \mathcal{A}_{234} \subset \ldots \subset \mathcal{A}_5 := \mathcal{A}_0 \sqcup \mathcal{A}_{234} \sqcup \cdots \sqcup \mathcal{A}_{1234}.$$

Homotopy decompositions and symplectic stabilizers

Proposition

The space $\mathcal{A}(\widetilde{\omega}_{\mathbf{c}}, \Sigma)$ is a submanifold of $\mathcal{A}(\widetilde{\omega}_{\mathbf{c}})$. The stratum $\mathcal{A}_0(\Sigma)$ is open and dense in $\mathcal{A}(\widetilde{\omega}_{\mathbf{c}}, \Sigma)$. If non-empty, the stratum $\mathcal{A}_{ijk}(\Sigma)$ is a codimension 2 submanifold. Moreover, in the case $N = 4$, the stratum $\mathcal{A}_{1234}(\Sigma)$ is a submanifold of codimension 4. $\square$

Proposition

The group $\mathrm{Diff}_h(\widetilde{M}_{\mathbf{c}}, \Sigma)$ acts on $\mathcal{A}(\widetilde{\omega}_{\mathbf{c}}, \Sigma)$ preserving the stratification. The stratum $\mathcal{A}_0$ is homotopy equivalent to the orbit of an integrable complex structure J_0. If non-empty, the stratum $\mathcal{A}_{ijk}$ is a codimension 2 submanifold that is homotopy equivalent to the orbit of an integrable complex structure J_{ijk}. The same is true for the stratum $\mathcal{A}_{1234}$. In all cases, the stabilizer is a group of complex automorphisms whose identity component retracts onto its maximal compact subgroup of Kähler isometries. $\square$

Example: Homotopy decompositions for $n = 3$

Taking the homotopy orbits of $\mathrm{Diff}(\widetilde{M}_{\mathbf{c}}, \Sigma)$, the inclusion $\mathcal{A}_0 \subset \mathcal{A}_0 \sqcup \mathcal{A}_{123}$ induces an inclusion

$$\mathrm{BSymp}(\widetilde{M}_{\mathbf{c}}, \Sigma) \subset \mathrm{BSymp}(\widetilde{M}_{\mathbf{c}'}, \Sigma)$$

whenever $\mathbf{c} \prec \mathbf{c}'$.

Consider the pushout diagram of inclusions

$$\begin{tikzcd} \mathcal{N}(\mathcal{A}_{123}) \setminus \mathcal{A}_{123} \arrow[r] \arrow[d] & \mathcal{N}(\mathcal{A}_{123}) \arrow[d] \\ \mathcal{A}_0 \arrow[r] & \mathcal{A}_0 \sqcup \mathcal{A}_{123} \end{tikzcd}$$

Up to homotopy, $\mathrm{Diff}(\widetilde{M}_{\mathbf{c}}, \Sigma)$ acts transitively on the stratum $\mathcal{A}_0$ with stabilizer $\mathbf{T}^2$ and on the stratum $\mathcal{A}_{123}$ with stabilizer $\mathbf{S}^1$. Since the stratum $\mathcal{A}_{123}$ has codimension 2 in $\mathcal{A}_0$, the link is a circle C_{123}.

Lemma

The link C_{123} is homotopic to the orbit of an almost complex structure $J_0 \in \mathcal{A}_0$ under the action of the stabiliser $\mathbf{S}^1$.

Example: Homotopy decompositions for $n = 3$

Applying the Borel construction with respect to the $\mathrm{Diff}(\widetilde{M}_{\mathbf{c}}, \Sigma)$ action, this gives

$$\begin{tikzcd} B\mathbb{1} \arrow[r] \arrow[d] & B\mathbf{S}^1 \arrow[d] \\ \mathrm{BStab}_0 \simeq B\mathbf{T}^2 \arrow[r] & B\mathbf{T}^2 \vee B\mathbf{S}^1 \end{tikzcd} \tag{9}$$

where BStab_0 is the the classifying space of $\mathrm{Symp}(\widetilde{M}_{\mathbf{c}}, \Sigma)$ when $c_1 + c_2 + c_3 \leq 1$. Thus the homotopy type of the stabilizer $\mathrm{BSymp}(\widetilde{M}_{\mathbf{c}}, \Sigma)$ when $c_1 + c_2 + c_3 < 1$ is given by $B\mathbf{T}^2 \vee B\mathbf{S}^1$.

Homotopy decompositions of $\mathsf{Conf}_3(\mathbf{CP}^2)$

The configuration space $F = \mathsf{Conf}_3(\mathbf{CP}^2)$ decomposes into two disjoint strata

$$F = F_0 \sqcup F_{123}$$

where

$$\begin{aligned} F_0 &= \{P \in F \mid \text{points of } P \text{ are in general position}\} \\ F_{123} &= \{P \in F \mid \text{points of } P \text{ are colinear}\} \end{aligned}$$

The stratum F_0 is open and dense in F, while F_{123} has codimension 2. The group $\mathrm{PGL}(3, \mathbb{C})$ of holomorphic automorphisms acts transitively on each stratum with stabilisers homotopy equivalent to Kähler actions $\mathbf{T}^2$ and $\mathbf{S}^1$, respectively, so that $F_0 \simeq \mathrm{PGL}(3)/\mathbf{T}^2$ and $F_{123} \simeq \mathrm{PGL}(3)/\mathbf{S}^1_{123}$. We can then apply the Borel construction to F relatively to the $\mathrm{PGL}(3, \mathbb{C})$ action to the following diagram

$$\begin{tikzcd} \mathcal{N}_{123} \setminus F_{123} \arrow[r] \arrow[d] & \mathcal{N}_{123} \arrow[d] \\ F_0 \arrow[r] & F \end{tikzcd}$$

where $\mathcal{N}_{123}$ is an invariant neighborhood of F_{123} in F.

Homotopy decompositions of $\mathrm{Conf}_3(\mathbf{CP}^2)$

We obtains the following pushout diagram

$$\begin{tikzcd} * \arrow[r] \arrow[d] & B\mathbf{S}^1_{123} \arrow[d] \\ B\mathbf{T}^2 \arrow[r] & B\mathbf{S}^1 \vee B\mathbf{T}^2 \end{tikzcd}$$

As this action is homotopy equivalent to the action of $\mathrm{Diff}_{[\mathbf{c}]}(\widetilde{M}_{\mathbf{c}}, \Sigma)$ on the stratification of $\mathcal{A}_{\mathbf{c}}(\Sigma)$, we recover the homotopy equivalences stated previously. This shows that the configuration space F is a finite dimensional model of the embedding space of 3 symplectic balls in $\mathbf{CP}^2$ in which the location of points corresponds to the capacities of the symplectic balls.

The main result

Proposition

For $n \leq 4$, the pushout diagrams coming from the symplectic stratification of $\mathcal{A}_{\mathbf{c}}(\Sigma)$ are homotopy equivalent to the ones coming from the complex stratification of $\mathrm{Conf}_n(\mathbf{CP}^2)$.

Claim: This also holds for $n = 5$.

Naive conjecture: Should hold for $n \leq 8$ as well.

The key result for $n \leq 4$

The main point is to show:

Proposition

The group $\mathrm{Diff}_h(\widetilde{M}_{\mathbf{c}}, \Sigma)$ *acts on* $\mathcal{A}(\widetilde{\omega}_{\mathbf{c}}, \Sigma)$ *preserving the stratification. The stratum* $\mathcal{A}_0$ *is homotopy equivalent to the orbit of an integrable complex structure* J_0*. If non-empty, the stratum* $\mathcal{A}_{ijk}$ *is a codimension* 2 *submanifold that is homotopy equivalent to the orbit of an integrable complex structure* J_{ijk}*. The same is true for the stratum* $\mathcal{A}_{1234}$*. In all cases, the stabilizer is a group of complex automorphisms whose identity component retracts onto its maximal compact subgroup of Kähler isometries.* □

To prove this, we have to understand how $\mathrm{Symp}_h(\widetilde{M}_{\mathbf{c}}, \Sigma)$ acts on spaces of symplectic configurations of embedded spheres that characterize each stratum in $\mathcal{J}_{\mathbf{c}}(\Sigma)$ and $\mathcal{A}_{\mathbf{c}}(\Sigma)$. The main questions are:

1. Show that the action is transitive for each configuration type.
2. Compute the stabilizer for each configuration type.

Configurations of type S_0 for $n = 3$

Configuration characterizing the stratum $\mathcal{J}_0(\Sigma)$ for $n = 3$.

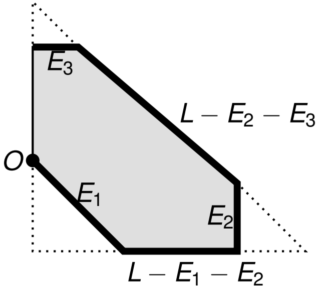


Figure: Moment image of the toric configuration S_0 for $n = 3$.

Its complement is a symplectically star-shaped toric domain.

Configurations of type S_{123} for $n = 3$

Configuration characterizing the stratum $\mathcal{J}_{123}(\Sigma)$ for $n = 3$.

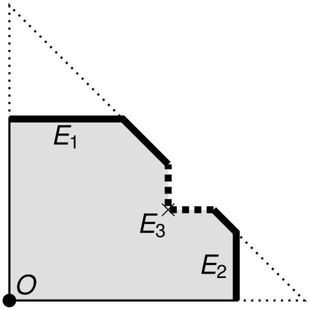


Figure: An almost toric construction of the configuration S_{123} for $n = 3$.

Its complement is a symplectically star-shaped toric domain.

Configurations of type S_0 for $n = 4$

Configuration characterizing the stratum $\mathcal{J}_0(\Sigma)$ for $n = 4$.

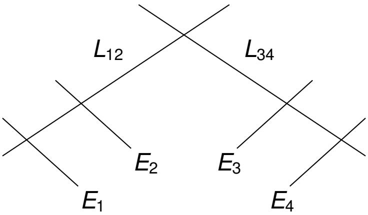


Figure: Configuration S_0 for $n = 4$

Its complement is diffeomorphic to the complement of two lines in $\mathbf{CP}^2$.

Configurations of type S_{ijk} for n=4

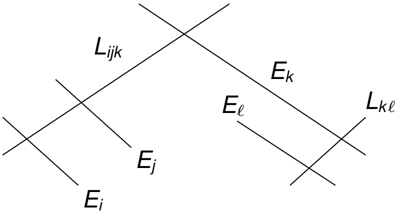


Figure: Configuration S_{ijk} for $n = 4$

Its complement is also diffeomorphic to the complement of two lines in $\mathbf{CP}^2$.

Configurations of type S_{ijk} for $n = 4$

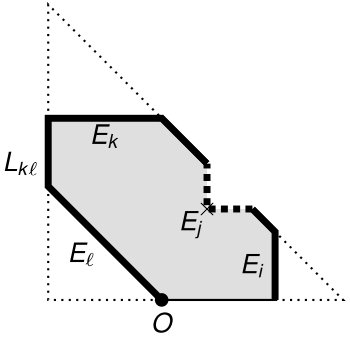


Figure: An almost toric construction of the configuration S_{ijk} for $n = 4$.

Its complement is a symplectically star-shaped toric domain when $c_j < 1 - c_i - c_\ell$ or $c_j < 1 - c_k - c_\ell$.

Configurations of type S_{1234}

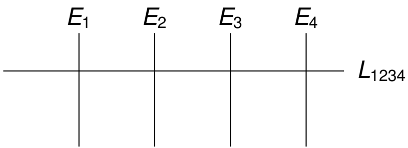


Figure: Configuration S_{1234}

Its complement is diffeomorphic to a ball.

Configurations of type S_{1234}

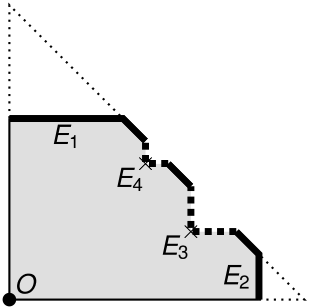


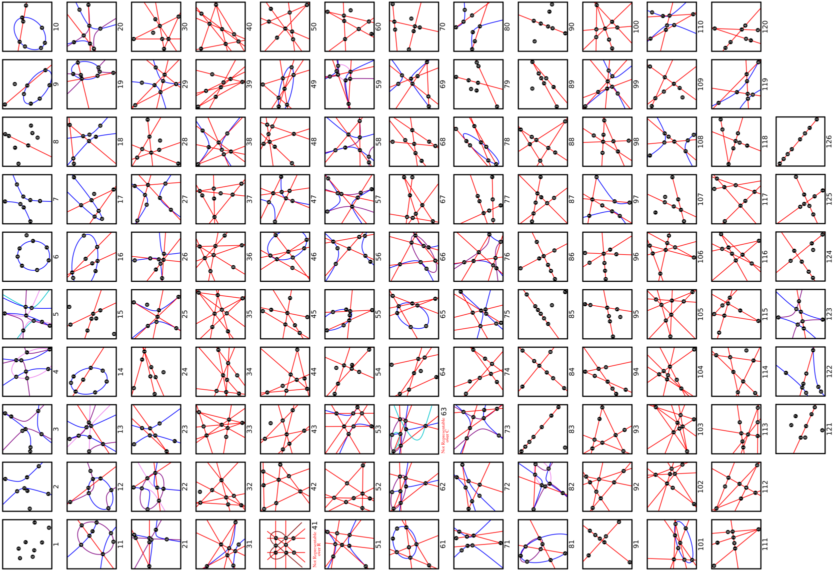
Figure: An almost toric construction of the configuration S_{1234}.

Its complement is a symplectically star-shaped toric domain.

Our current knowledge

- (Hind-Ivrii, 2010) In $S^2 \times S^2$ and $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$, any two homologous, embedded symplectic spheres of negative self-intersection are Hamiltonian isotopic.
- (Borman-Li-Wu, 2014) For $n \geq 3$, $\mathsf{Symp}_h(M_{\mathbf{c}}, \widetilde{\omega}_{\mathbf{c}})$ acts transitively on all homologous, symplectic, embedded spheres with $-4 \leq [S] \cdot [S] \leq -2$ as long as $[S]$ is not characteristic (e.g. $L - E_1 - \cdots E_n$ is characteristic).
- (Anjos-Kedra-P, 2023) Same holds for $[S]$ characteristic by a simple modification of the argument.
- (McDuff-Opshtein, 2015) If $-4 \leq [S] \cdot [S] \leq -2$, and if $GW([A]) \neq 0$, then the space $\mathcal{J}([A], S)$ is connected.
- It follows that if the configurations caracterising the strata of $\mathcal{A}_{\mathbf{c}}(\Sigma)$ only contain a single sphere with $-4 \leq [S] \cdot [S] \leq -2$ and if the other components have $GW([A]) \neq 0$, then $\mathsf{Symp}_h(\widetilde{M}_{\mathbf{c}}, \widetilde{\omega}_{\mathbf{c}})$ acts transitively on the orthogonal symplectic configurations.
- If the complement $(\widetilde{M}_{\mathbf{c}} \setminus C)$ of a configuration C is nice enough, then $\mathsf{Symp}_{\mathrm{comp}}(\widetilde{M}_{\mathbf{c}} \setminus C)$ is contractible, and the homotopy type of the symplectic stabiliser of C can be computed from local data near C. This is the case for all configurations that occur for $n \leq 4$.

The 126 relative positions of 8 points in $\mathbf{CP}^2$



Rendering by Taylor Brysiewicz