Symplectic embeddings into disk cotangent bundles of spheres

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Good (but hard) questions

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The first obstruction is the volume: $\mathrm{Vol}(M_1) \leq \mathrm{Vol}(M_2)$, are there others?

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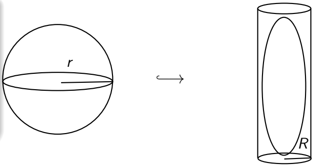

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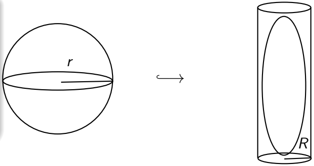

Symplectic embeddings $\neq$ Volume preserving embeddings

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- $c(B^{2n}(r), \omega_0) > 0$ and $c(Z^{2n}(r), \omega_0) < \infty$.

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- ECH capacities c_k^{ECH} (Hutchings) - only in dimension 4.

Examples of ECH capacities

- Ellipsoids (Hutchings):

$$E(a, b) = \left\{ (z_1, z_2) \in \mathbb{C}^2 \;\middle|\; \pi\left(\frac{|z_1|^2}{a} + \frac{|z_2|^2}{b}\right) < 1 \right\}.$$

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  $$(c_k(B(a), \omega_0))_k = (0, a^{\times 2}, 2a^{\times 3}, 3a^{\times 4}, 4a^{\times 5}, \dots).$$

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It agrees with the "even multiples" that appears in the sequence for the ball $B(\ell)$.

Toric domains

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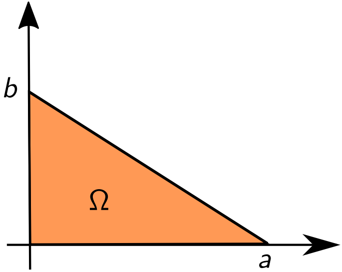

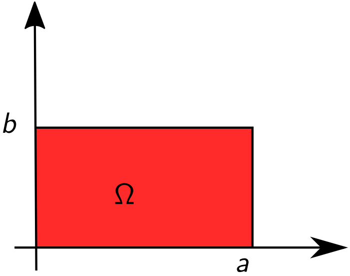

The Arnold-Liouville theorem

Fix (M^{2n}, ω) and let $F = (H^1, \ldots, H^n) : M \to \mathbb{R}^n$ whose components Poisson commute, i.e., $\{H_i, H_j\} := \omega(X_{H_i}, X_{H_j}) = 0$.

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  Then there exists a symplectomorphism $\Phi : (U, \omega) \to (\phi(U) \times \mathbb{T}^n, \omega_0)$ such that the following diagram commutes.

$$\begin{array}{ccc} U & \xrightarrow{\Phi} & \phi(U) \times \mathbb{T}^n \cong \mathbb{X}_{\phi(U)} \\ \downarrow F & & \downarrow \pi_1 \\ F(U) & \xrightarrow{\phi} & \phi(U) \end{array}$$

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- *The Lagrangian product of a hypercube and a symmetric region in* $\mathbb{R}^{2n}$ *is symplectomorphic to a toric domain. (Ramos, Sepe, 2019)*

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Reeb vector field R_λ on (S^*N, λ) is dual to the geodesic vector field on SN via g^b: $TN \to T^*N$. Moreover, the action $\mathcal{A}(\gamma) = \int_\gamma \lambda$ of a Reeb orbit γ on S^*N agrees with the length of the projected geodesic on the base N.

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Integrable systems in spheres of revolution

Theorem (F., Ramos, Vicente)

Let $S \subset \mathbb{R}^3$ *be a sphere of revolution with a unique equator.*

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Nested domains $H_\varepsilon^{-1}([0,1)) \cong X_{\Omega_\varepsilon}$ converging to $D^*(S \setminus \{P_N\}) \cong \operatorname{int} X_{\Omega_S}$ when $\varepsilon \to 0$. □

Toric image

In fact, if S is obtained revolving the graph of a function u around the z-axis, $\Omega_S \subset \mathbb{R}^2_{\geq 0}$ given in the Theorem is the region bounded by the coordinate axis and the curve parametrized by

$$\begin{cases} (f_S(j), f_S(j) + j), \text{ if } 0 \leq j \leq 2\pi u(z_0), \\ (f_S(-j) - j, f_S(-j)), \text{ if } -2\pi u(z_0) \leq j \leq 0, \end{cases}$$

for the function

$$f_S(j) = 2 \int_{z_-(1,j)}^{z_+(1,j)} \sqrt{\left(1 - \frac{j^2}{4\pi^2 u(z)^2}\right)(u'(z)^2 + 1)}\, dz.$$

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Here z_0 is the unique critical point of u and $z_\pm(1,j)$ are the solutions of $(2\pi u(z))^2 - j^2 = 0$. It follows that $f_S(0) = L$ coincides with the length of the meridians and $f_S(2\pi u(z_0)) = 0$.

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If $S \subset \mathbb{R}^3$ is a Zoll sphere of revolution, then $\mathbb{X}_{\Omega_S}$ is the symplectic bidisk $P(\ell, \ell)$, where ℓ is the length of any simple closed geodesic on S.

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i.e., $\Omega_S = [0, \ell] \times [0, \ell]$. □

Gromov width of D^*S_{Zoll}

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On other hand, if $(B(a), \omega_0) \hookrightarrow (D^*S, \omega_{can})$, we have

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It yields $c_{Gr}(D^*S, \omega_{can}) \leq \ell$. $\square$

Ellipsoids of revolution

For $a, b, c > 0$, let $\mathcal{E}(a, b, c) \subset \mathbb{R}^3$ be the ellipsoid defined by the equation:

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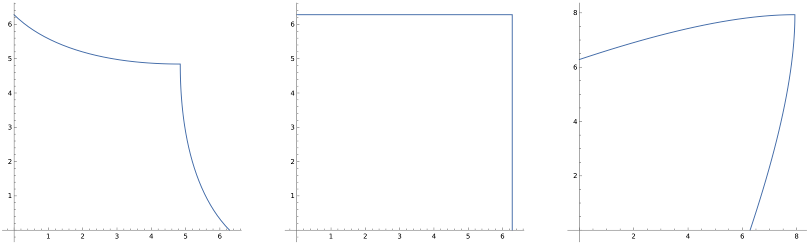

Figure: The region $\Omega_{\mathcal{E}(1,1,c)}$ for $c = 0,5$; $c = 1$; $c = 1,5$, respectively.

Gromov width of $D^*\mathcal{E}(1,1,c)$

Theorem (F., Ramos, Vicente)

The Gromov width of $D^*\mathcal{E}(1,1,c)$ is given by

$$c_{Gr}(D^*\mathcal{E}(1,1,c),\omega_{can}) = \begin{cases} \alpha(c), \text{ for } 0 < c < 1/2, \\ 2\pi, \text{ for } 1/2 \leq c \leq 1, \\ \beta(c), \text{ for } 1 < c < c_0, \\ 4\pi, \text{ for } c \geq c_0. \end{cases}$$

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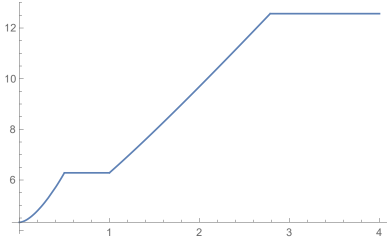

Figure: Graph of function $c \mapsto c_{Gr}(D^*\mathcal{E}(1,1,c),\omega_{can})$.

A different embedding problem - comparing metrics on S^2

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Note that we have the upper bound

$$R_0 = \max_{\nu \in D_g^* S^2} \sqrt{g_0(\nu, \nu)} = \max_{\|\nu\|_g = 1} \sqrt{g_0(\nu, \nu)}$$

obtained by the inclusion.

A computation and immediate consequence

Theorem (F.)

Let (S^2, g) *be a Riemannian sphere such that* $1/4 < K \leq 1$*, where* K *is the sectional curvature. Hence*

$$c_1(D_g^* S^2, \omega_{can}) = 2L, \qquad (1)$$

where L *is the length of a shortest closed geodesic for* g*. Moreover, it is well known that* $L \in [2\pi, 4\pi)$ *in this case.*

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Corollary

Let (S^2, g) be a Riemannian sphere such that $1/4 < K \leq 1$. The existence of a symplectic embedding

$$(D_g^* S^2, \omega_{can}) \hookrightarrow (D_{g_0}^* S^2(r), \omega_{can}),$$

forces the inequality $L \leq 2\pi r$. In particular, $L \leq 2\pi R_0$.

Systolic inequalities

Some related results:

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- **(Rotman 2005):** $L \leq 4diam$ for any Riemmanian metric;
- **(Adelstein, Pallete 2020):** $L \leq 3diam$ for Riemannian metrics with $K \geq 0$.

Thank you!