Quantum Groups & Digital setting

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Quantum Groups & Hopf algebras

- Hopf algebras - nice objects - axioms in 1940 [Heinz Hopf]
- Special examples of Hopf algebras (1980s): Quantum Groups in integrable systems $\longrightarrow$ then theory was formalized by Drinfel'd, Jimbo, Reshetikin, Takhtajan, Fadeev etc.

- Quantum Gravity research: Quantum Groups as deformed symmetries of noncommutative spaces (NCG)

What is 'quantum' in Quantum Groups?

Basic concepts: States & Observables

Classical Mechanics		Quantum Mechanics
M - phase space [Poisson mfld]	$\rightsquigarrow$	V - Hilbert space
states: points of M		states: 1 dim subspaces of V
observables: $F(M) = C^\infty(M)$	$\rightsquigarrow$	observables: $Op(V)$

observables form an assoc. algebra

commutative $\xRightarrow{\text{quantization}}$ noncommutative

Idea by Moyal'1949 – change the product in $F(M)$

Classical	Quantum
$(F(M), \cdot)$	$(F(M), *_h) = F_h(M)$

$F_h(M)$-quantization of $F(M)$

$F_h(M)$ – noncommutative algebra of functions $\rightsquigarrow$
$\rightsquigarrow$ Realm of A. Connes NCG

A. Connes $\longrightarrow$ noncommutative spaces
(Space is determined by the algebra of functions on it
$\Rightarrow$ noncommutative algebra should be viewed as the space of functions on noncomm. space)

Main idea of non-commutative geometry (NCG)

M, $F(M) = C^\infty(M)$ $\rightsquigarrow$ A – abstract algebra [quantum space]

G – symmetry (Lie group) $\rightsquigarrow$ H- Quantum Group (Hopf algebra)

$\Omega^1(M)$-differential calculus $\rightsquigarrow$ $(\Omega^1(M), d)$ A - A bimodule of 'algebraic' 1-forms

$\Downarrow$

Riem. geom.

UNITAL ASSOCIATIVE ALGEBRA (over k)

is given by the triple

(A, μ, i) where

- A - vector space
- $\mu : A \otimes A \to A$
- $i : k \to A$

are linear maps satisfying the axioms:

(Associativity)

the square

$$\begin{tikzcd}
\overset{abc}{A \otimes A \otimes A} \arrow[r, "\mu \otimes id"] \arrow[d, "id \otimes \mu"] & \overset{(ab)c}{A \otimes A} \arrow[d, "\mu"] \\
\underset{a(bc)}{A \otimes A} \arrow[r, "\mu"] & A
\end{tikzcd} \qquad (A1)$$

commutes.

(Unit)

The diagram

$$\begin{tikzcd}
k \otimes A \arrow[r, "i \otimes id"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] & A \arrow[l, "id \otimes i"'] \arrow[dl, "\cong"] \\
 & A &
\end{tikzcd} \qquad (A2)$$

$i(1) = 1_A$

commutes.

COUNITAL COASSOCIATIVE COALGEBRA (over k)

is given by the triple

(C, Δ, ε) where

- C - vector space
- $\Delta : C \to C \otimes C$ coproduct (comultiplication)
- $\varepsilon : C \to k$ (counit)

are linear maps satisfying:

(Coassoc.)

$$(HA1) \qquad \begin{tikzcd}
C \otimes C \otimes C & C \otimes C \arrow[l, "\Delta \otimes id"'] \\
C \otimes C \arrow[u, "id \otimes \Delta"] & C \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}$$

(counit)

$$(HA2) \qquad \begin{tikzcd}
k \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes id"'] \arrow[r, "id \otimes \varepsilon"] & C \otimes k \\
 & C \arrow[u, "\Delta"'] \arrow[ul, "\cong"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

We get the def. of coalgebra by reversing all the arrows in the def. of algebra!

Note: for a finite dim. alg. A

$$C = A^*, \quad \Delta = \mu^*$$
$$\varepsilon = i^*$$

Def. Hopf algebra $(H, \mu, i, \Delta, \varepsilon, S)$ is

a BIALGEBRA, i.e.

1. H is a (unital, assoc.) algebra (H, μ, i) satisfying (A1), (A2).

2. H is a (coun., coassoc.) coalgebra (H, Δ, ε) satisfying (HA1), (HA2).

in a compatible way:

3. Δ, ε are algebra homomorphisms

$$\begin{aligned} \Delta(ab) &= \Delta(a)\Delta(b) \\ \varepsilon(ab) &= \varepsilon(a)\varepsilon(b) \end{aligned} \quad \Bigg| \quad \begin{aligned} \Delta(1) &= 1 \otimes 1 \\ \varepsilon(1) &= 1 \end{aligned}$$

which is equipped with

ANTIPODE MAP $S : H \to H$ s.t.

$$\begin{tikzcd}
H \otimes H \arrow[r, "S \otimes id"] & H \otimes H \arrow[d, "\mu"] & H \otimes H \arrow[l, "id \otimes S"'] \\
H \arrow[u, "\Delta"] \arrow[r, "i \circ \varepsilon"] & H & H \arrow[u, "\Delta"'] \arrow[l, "i \circ \varepsilon"']
\end{tikzcd}$$

H.A. is commutative if H is comm. as an algebra

H.A. is co-commutative if H is co-comm. as an coalgebra

$$\Delta^{op} = \Delta$$

$$\Delta^{op} = \tau \circ \Delta$$

EXAMPLES:

1. $\mathbb{C}G$, $\Delta(g) = g \otimes g$, $\varepsilon(g) = \overset{e}{1}$, $S(g) = g^{-1}$

2. $U(\mathfrak{g})$, $\mathfrak{g}$ - Lie alg.

$$\left.\begin{aligned} &\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \varepsilon(X) = 1, \quad S(X) = -X \\ &X \in \mathfrak{g} \end{aligned}\right\} \rightsquigarrow$$

can be deformed by e.g. Drinfeld twist

In general,

- Let x^μ be a basis of our (unital associative) algebra A with $x^0 = 1$ the unit and $\mu = 0, \cdots, n-1$. We write structure constants by

$$x^\mu x^\nu = V^{\mu\nu}{}_\rho x^\rho, \quad V^{\mu\nu}{}_\rho \in k.$$

$$V^{0\mu}{}_\nu = \delta^\mu_\nu = V^{\mu 0}{}_\nu, \quad V^{\rho\nu}{}_\lambda V^{\lambda\mu}{}_\gamma = V^{\nu\mu}{}_\lambda V^{\rho\lambda}{}_\gamma.$$

- If A admits the bialgebra structure, then we express the coproduct in terms of structure constants as,

$$\Delta x^\mu = C^\mu{}_{\nu\rho} x^\nu \otimes x^\rho, \quad C^\mu{}_{\nu\rho} \in k, \quad \epsilon(x^\mu) = \epsilon^\mu \in k.$$

For the Hopf algebra one also has the antipode

$$S x^\mu = s^\mu{}_\nu x^\nu, \quad s^\mu{}_\nu \in k.$$

- An algebra homomorphism $\phi(x^\mu) = \phi^\mu{}_\nu x^\nu$ from an algebra with product V to one with product V' means

$$V\phi = (\phi \otimes \phi)V', \quad \boxed{V^{\mu\nu}{}_\rho \phi^\rho{}_\tau = \phi^\mu{}_\alpha \phi^\nu{}_\beta V'^{\alpha\beta}{}_\tau.}$$

and we also demand that $\eta_\mu \phi^\mu{}_\nu = \eta'_\nu$ for the units (if both algebras are in standard form then this is $\phi^0{}_\nu = \delta^0{}_\nu$). If ϕ is surjective (such as an isomorphism) then this unit condition is automatic.

- Coalgebra homomorphism $\psi(x^\mu) = \psi^\mu{}_\nu x^\nu$ from a coalgebra with coproduct C' to one with coproduct C means

$$C'(\psi \otimes \psi) = \psi C, \quad \boxed{C'^\tau{}_{\alpha\beta} \psi^\alpha{}_\mu \psi^\beta{}_\nu = \psi^\tau{}_\rho C^\rho{}_{\mu\nu}.}$$

In "digital" setting:

$A = \{x^\mu\}$, $V^{\mu\nu}{}_\rho \in \{0,1\}$

$C^\mu{}_{\nu\rho} \in \{0,1\}$

Digital Quantum Groups

based on the joint work with S. Majid, JMP 61, 103510 (2020) [arXiv:2006.16799]

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- Aim: complete classification of all Hopf algebras and bialgebras up to dimension $n \leq 4$, working over the field $\mathbb{F}_2 = \{0, 1\}$ of two elements.

- The starting point: classification of algebras [commutative algebras over $\mathbb{F}_2$ in low dimensions were already classified in our previous work *S. Majid, A.P., J.Phys.A (2019) arXiv:1807.08492*]
- when we also consider noncommutative algebras, we obtained:

1. for $\mathrm{n} = 2$, there are 3 commutative A, B, C and none noncommutative
2. For $\mathrm{n} = 3$, there are 6 commutative A - F, and one noncommutative G
3. For $\mathrm{n} = 4$, there are 16 commutative A - P, and 9 noncommutative NA - NI

Briefly on results

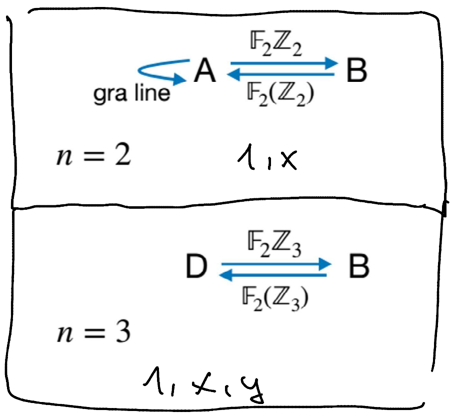
- We succeeded in determining all inequivalent bialgebras and Hopf algebras of dimension $n \leq 4$ over $\mathbb{F}_2$ and presented our results in the form of extended graphs.

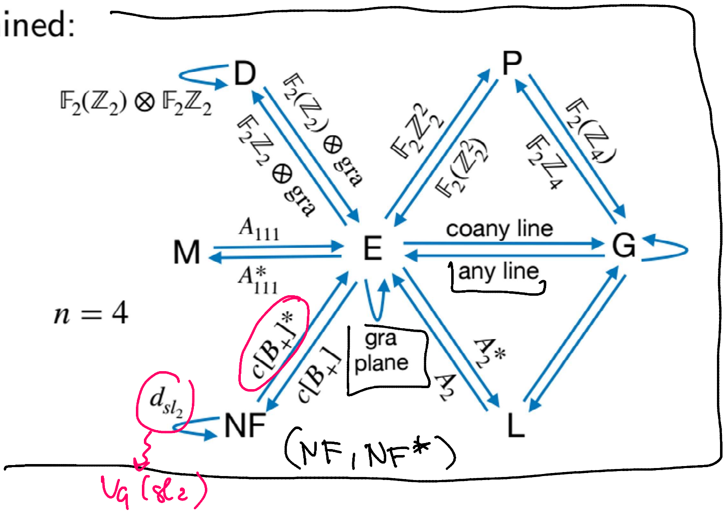
- We can represent our results as a quiver by drawing an arrow for each bialgebra of Hopf algebra according to its type.

- For example, $\boxed{\mathrm{A} \to \mathrm{B}}$ means a Hopf algebra with algebra A and coalgebra isomorphic to the dual of B, i.e. the type (A, B^*).

Example $\mathbb{C}G$

For Hopf algebras, we obtained:





- We identified **new Hopf algebras** of dimension $n = 4$ over $\mathbb{F}_2$.

$\to d_{sl_2}$

$c[B_+]^*$

gra = Grasmann line

$\mathbb{F}_2[x]/\langle a(x)\rangle$

a(x) – includes terms with powers of 2

n=3 case

any = anyonic line

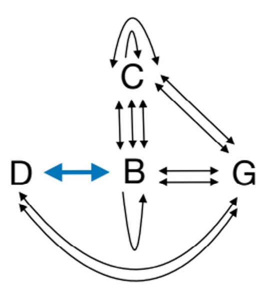
$x^4 = 0$

- In 3 dimensions have 6 commutative algebras and 1 noncommutative one (A-G) over $\mathbb{F}_2$.

A : $x^2 = y^2 = xy = 0$ (the unital algebra with all other products zero).
B : $x^2 = x, y^2 = y, xy = 0$ (this is the algebra of $\mathbb{F}_2(\mathbb{Z}_3)$ or functions on a triangle).
C : $x^2 = x, y^2 = xy = 0$ (this is $\mathbb{F}_2[z]/\langle z^3 + z\rangle$, with $z = 1 + x + y$ or conversely $x = 1 + z^2$ and $y = z + z^2$).
D : $x^2 = y, y^2 = x, xy = x + y$ (this is the group algebra $\mathbb{F}_2\mathbb{Z}_3 = \mathbb{F}_2[z]/\langle z^3 + 1\rangle$, with $z = 1 + x$).
E : $x^2 = y, y^2 = xy = 0$ (this is $\mathbb{F}_2[x]/\langle x^3\rangle$, the anyonic line).
F : $x^2 = y, xy = 1 + y, y^2 = 1 + x + y$ (this is the field $\mathbb{F}_8 = \mathbb{F}_2[x]/\langle x^3 + x^2 + 1\rangle$).
G : $x^2 = x, y^2 = 0, xy = y, yx = 0$ (this is noncommutative but G≅G^{op} by $x \mapsto 1 + x$ and $y \mapsto y$).

- Only $B = \mathbb{F}_2(\mathbb{Z}_3)$ and $D = \mathbb{F}_2\mathbb{Z}_3$ admit a Hopf algebra structure (namely the unique one indicated by the notation as group algebra or function algebra on a group).
- The algebras B,C,D,G admit many bialgebras (but no further Hopf algebras) and the algebras A,E,F admit no bialgebra structures.

n=3 BIALGEBRAS:
(black arrows)



In the Hopf algebra (as opposed to bialgebra) version we have only

$$\begin{tikzcd} D \arrow[r, shift left, "\mathbb{F}_2\mathbb{Z}_3"] & B \arrow[l, shift left, "\mathbb{F}_2(\mathbb{Z}_3)"] \end{tikzcd}$$

$\mathbb{F}_2\mathbb{Z}_3$ is H.A. (D, B^*)
$\mathbb{F}_2(\mathbb{Z}_3)$ is H.A. (B, D^*)

$n = 3$

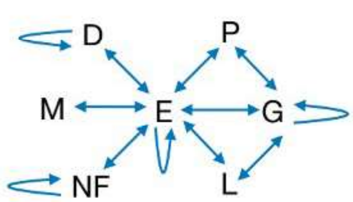
For example: ALGEBRA D:

D : $x^2 = y, y^2 = x, xy = x + y$ (this is the group algebra $\mathbb{F}_2\mathbb{Z}_3 = \mathbb{F}_2[z]/\langle z^3 + 1\rangle$, with $z = 1 + x$).

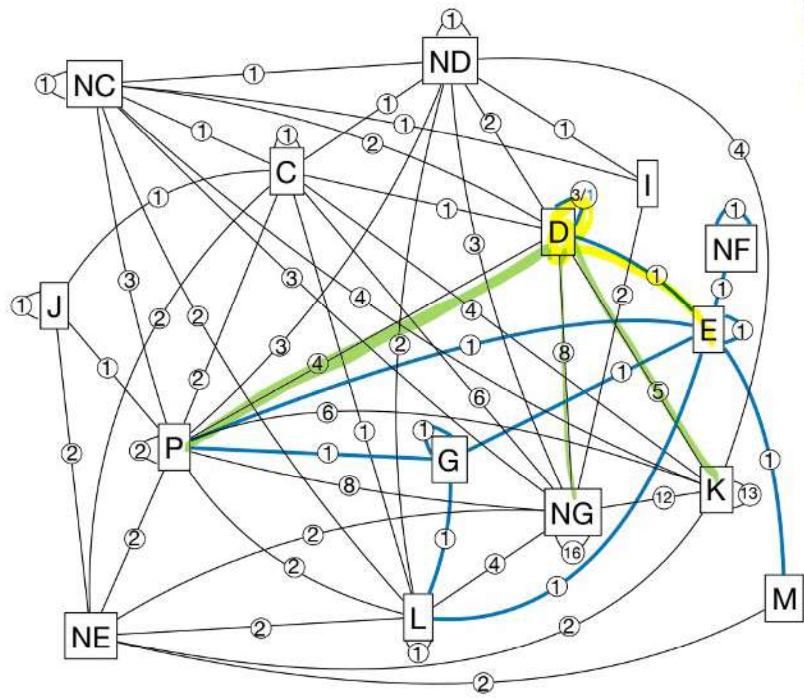
D.1. (Hopf algebra) $\Delta x = 1 \otimes x + x \otimes 1 + x \otimes x$, $\Delta y = 1 \otimes y + y \otimes 1 + y \otimes y$,
$\epsilon x = 0 = \epsilon y$, $Sx = y$, and $Sy = x$.
Dual is commutative algebra B with $1 = y_0$, $y_1^2 = y_1$, $y_1 y_2 = 0$, $y_2^2 = y_2$.
D.2. $\Delta x = 1 \otimes x + x \otimes 1 + x \otimes x + y \otimes x$, $\Delta y = 1 \otimes y + y \otimes 1 + x \otimes y + y \otimes y$, and $\epsilon x = 0 = \epsilon y$.
Dual is noncommutative algebra G with $1 = y_0$, $y_1^2 = y_1$, $y_1 y_2 = y_2$, $y_2 y_1 = y_1$, and $y_2^2 = y_2$.
D.3. $\Delta x = 1 \otimes x + x \otimes 1 + x \otimes x + x \otimes y$, $\Delta y = 1 \otimes y + y \otimes 1 + y \otimes x + y \otimes y$, and $\epsilon x = 0 = \epsilon y$.
Dual is noncommutative algebra G with $1 = y_0$, $y_1^2 = y_1$, $y_1 y_2 = y_1$, $y_2 y_1 = y_2$, and $y_2^2 = y_2$.

n=4 case

- there are 16 unital commutative algebras A - P and 9 noncommutative ones NA - NI
- several are known to have at least one or two commutative and cocommutative Hopf algebra structures, so part of our work was to **identify known Hopf algebras** and check that all of them turn up.
- the basis elements x^μ explicitly as $1, x, y, z$,



- The vertices here are the $n = 4$ algebras, with NF the only noncommutative one.
- There is just **one Hopf algebra which is both noncommutative and noncocommutative**, namely the self-arrow on NF
- The full picture for all bialgebras is also found but has too many arrows to draw as a quiver, so this is presented instead as an extended weighted graph.



(D, D^*) – Hopf algebra

(D, E^*) – Hopf algebra

$(D, P^*)_1$

$(D, P^*)_2$

$(D, P^*)_3$

$(D, P^*)_4$

$(D, K)_{1\ldots 5}$

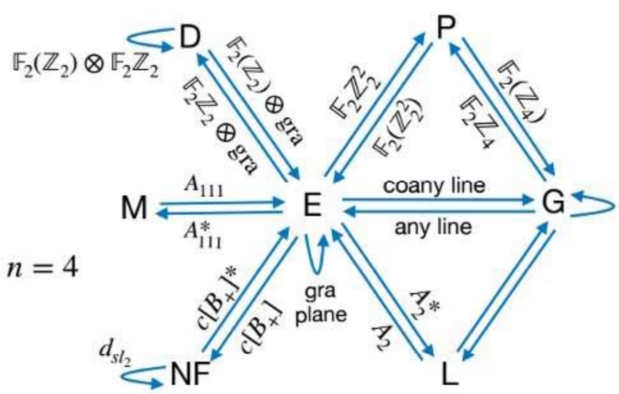
$(D, NG)_{1\ldots 8}$

BIALGEBRAS

- We identified all 6 possible tensor products of the three $n=2$ algebras $\mathbb{F}_2\mathbb{Z}_2$ (the $x^2=0$ (A) unital algebra for $n=2$), $\mathbb{F}(\mathbb{Z}_2), \mathbb{F}_4$ and found that only 5 of them are distinct, namely

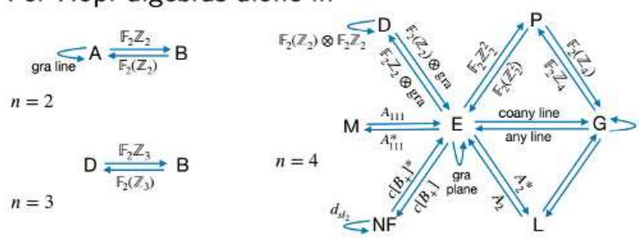
$$\mathrm{D} = \mathbb{F}_2(\mathbb{Z}_2) \otimes \mathbb{F}\mathbb{Z}_2, \quad \mathrm{E} = \mathbb{F}_2\mathbb{Z}_2 \otimes \mathbb{F}_2\mathbb{Z}_2, \quad \mathrm{H} = \mathbb{F}_4 \otimes \mathbb{F}_2\mathbb{Z}_2,$$

$$\mathrm{N} = \mathbb{F}_4 \otimes \mathbb{F}_4 \cong \mathbb{F}_2(\mathbb{Z}_2) \otimes \mathbb{F}_4, \quad \mathrm{P} = \mathbb{F}_2(\mathbb{Z}_2) \otimes \mathbb{F}_2(\mathbb{Z}_2).$$



Conclusions

- We succeeded in determining all inequivalent bialgebras and Hopf algebras of dimension $n \leq 4$ over $\mathbb{F}_2$.
- We presented our results in the form of extended graphs.
- For Hopf algebras alone in



we identified or described all.
- One important lesson is that while it is common practice to refer to an algebra by its most important role, for example $\mathbb{F}_2\mathbb{Z}_2$ for the group algebra of the group $\mathbb{Z}_2$, and we did the same when introducing our algebras for the first time, we now see that is much better to think of these *as labels of the arrows*, *not* of the vertices.
- We analysed the Fourier transform and quasitriangular structures

based on the joint work with S. Majid, JMP 61, 103510 (2020) [arXiv:2006.16799]

[S. Majid, A.P.
J. Math. Phys. 59, 033505 (2018); arXiv:1701.06919
J. Phys. A: Math. Theor. 53 115202 (2020); arXiv:1807.08492
J. Math. Phys. 61, 103510 (2020); arXiv:2006.16799.]

Riemannian geometry in digital setting

A

(Ω^1, d) : Ω^1 - A-A bimodule

$d : A \to \Omega^1$ satisfying Leibniz rule

$g \in \Omega^1 \otimes_A \Omega^1$

∇ , R_∇ . . . .

Definition

A first order differential calculus (Ω^1, d) over A means:

1. Ω^1 is an $A - A$ -bimodule
2. A linear map $\mathrm{d} : A \to \Omega^1$ such that

$$\mathrm{d}(xy) = (\mathrm{d}x)y + x\mathrm{d}y \quad , \forall x, y \in A$$

3. $\Omega^1 = span\{xdy\}, \quad x, y \in A$
4. (optional) $\ker d = k.1$ - connectedness condition

$1, x, y, z$

dx

dy

dz

Definition

DGA on an algebra A is:

1. A graded algebra $\Omega = \oplus_n \Omega^n$, $\Omega^0 = A$
2. $\mathrm{d} : \Omega^n \to \Omega^{n+1}$, s.t. $\mathrm{d}^2 = 0$ and

$$\mathrm{d}(\omega\rho) = (\mathrm{d}\omega) \wedge \rho + (-1)^n \omega \wedge \mathrm{d}\rho$$

$\forall \omega, \rho \in \Omega, \quad \omega \in \Omega^n.$

3. $A, \mathrm{d}A$ generate Ω
(optional surjectivity condition - if it holds we say it is an **exterior algebra** on A)

Hopf algebra acting on "quantum space" A:

$H \triangleright A$: $h \triangleright (ab) = \mu(h_{(1)}(a) \otimes h_{(2)}(b))$ [generalized Leibniz rule]

$\Delta(h) = h_{(1)} \otimes h_{(2)}$ coproduct in H

in classical case: $\Delta h = \underbrace{h \otimes 1 + 1 \otimes h}$

$\to h \triangleright (ab) = h(a) \cdot b + a \cdot h(b)$ [Leibniz rule]

Class. Geom. $\quad fg = gf$

$f dg = dg f$

$\downarrow$

NCG $\quad f dg \neq dg f \to$ generalization of "classical" differential geometry, even if $fg = gf$

$+ \underbrace{fg \neq gf} \Leftarrow$

$\downarrow$

Quantum Gravity