Lie theory in tensor categories with applications to modular representation theory.

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Let G be a finite group, p a prime. (the interesting case is when p divides $|G|$). Let V be a finite dimensional representation of G over an algebraically closed field k of characteristic p. Define

$d_n(V)$ = number of indecomposable summands in $V^{\otimes n}$ of dimension coprime to p.

It is clear that

$$d_{n+m}(V) \geq d_n(V)\, d_m(V).$$

and $$d_n(V) \leq (\dim V)^n.$$

<u>Lemma</u> (Fekete). If $\{a_n\}$ is a sequence of positive numbers such that $a_{n+m} \geq a_m a_n$

and $a_n \leq C^n$ for some $C > 0$
then $\exists \lim_{n \to \infty} \sqrt[n]{a_n} \in \mathbb{R}_{>0}$.

So we can define growth dimension
$$d(V) = \lim_{n \to \infty} d_n(V)^{1/n} \in \mathbb{R}_{\geq 0}.$$

Obvious properties:

1) $d(V \oplus W) \geq d(V) + d(W)$

2) $d(V \otimes W) \geq d(V) d(W)$

3) $d(X) = 0 \iff$ all indecomposable summands of X have dimension divisible by p. (such representations are called negligible and they form a tensor ideal in the category of representations).

4) $d(X) > 0 \Rightarrow 1 \leq d(X) \leq \dim_{\Bbbk} X$.

It turns out we can actually say a lot more.

Theorem (Coulembier – E – Ostrik, 2021)

1. $d(\bullet)$ extends to a character of the split Grothendieck ring (Green ring) of $\mathrm{Rep}_k(G)$. In other words,
$$d(V \oplus W) = d(V) + d(W)$$
and
$$d(V \otimes W) = d(V) \cdot d(W)$$

2. Let $q = e^{\frac{\pi i}{p}}$ and
$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}} = q^{m-1} + \dots + q^{1-m} \text{ for } m \in \mathbb{N}.$$
Then $\forall X \in \mathrm{Rep}_k(G)$, $d(X)$ is a linear combination of $[m]_q$, $1 \leq m \leq \frac{p}{2}$ with nonnegative integer coefficients. In particular, for $p = 2, 3$, $d(X)$ is an integer.

Remark: this holds for any (super) affine group scheme G/k

Example. Let $p = 5$, $G = \mathbb{Z}/5$, $V = J_3$ – the 3-dimensional indecomposable representation :

$$1 \in \mathbb{F}_5 \longmapsto \begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$$

$\overset{\shortparallel}{\mathbb{Z}/5}$

Then

$$V^{\otimes n} = a_n \cdot J_1 \oplus b_n \cdot J_3 \oplus c_n \cdot J_5$$

trivial

Since

$$V \otimes J_1 = J_3$$
$$V \otimes J_3 = J_1 \oplus J_3 \oplus J_5$$
$$V \otimes J_5 = 3 J_5$$

We get

$$a_{n+1} = b_n \Rightarrow a_n = b_{n-1}$$
$$b_{n+1} = a_n + b_n$$

$$\Rightarrow b_{n+1} = b_n + b_{n-1}$$ ← Fibonacci recursion.

$$d_n(V) = a_n + b_n = b_{n+1}$$

$$\Rightarrow d(V) = \frac{1+\sqrt{5}}{2} = [2]_q,$$

$$q = e^{\frac{\pi i}{5}}.$$

$\boxed{\text{In general in characteristic 5: } d(X) = r + s\frac{1+\sqrt{5}}{2},\ r,s \in \mathbb{Z}_{\geq 0}}$

Let me now explain the proof of this theorem. The proof is based on the theory of tensor categories.

Recall that a symmetric tensor category is a category $\mathcal{C}$ which has the structures and properties of the category of finite dimensional representations of a group. $\left(\begin{array}{c} k=\bar{k} \\ \text{field} \end{array}\right)$

+ • k-linear abelian
  • artinian: objects have finite length, $\dim \mathrm{Hom}(X,Y) < \infty$.

$\times$ • Monoidal: $\otimes$, associativity + pentagon, $\mathbb{1}$.
  • symmetric: $X \otimes Y \xrightarrow{c_{XY}} Y \otimes X$, hexagons, $c_{YX} \circ c_{XY} = 1$.
  • rigid: $X \mapsto X^*$, rigidity axioms

distrib. • $\otimes$ bilinear on morphisms
  • $\mathrm{End}(\mathbb{1}) = k$

Ex. 1. If G is a group (more generally, affine group scheme over k) $\mathrm{Rep}_k G$ is a STC. E.g. for $G=1$ we get the category Vec_k.

2. The category of supervector spaces $s\mathrm{Vec}_k$ (char $k \neq 2$):

$$s\mathrm{Vec}_k = \{ V = V_0 \oplus V_1 - \mathbb{Z}/2\text{-graded spaces} \},$$

$$c_{X,Y}(x \otimes y) = (-1)^{\deg x \cdot \deg y}(y \otimes x).$$

3. G - affine supergroup scheme over k (i.e. $\mathcal{O}(G)$ is a commutative Hopf algebra in $s\mathrm{Vec}_k$).

Let $z \in G(k)$, $z^2 = 1$, z acts on $\mathcal{O}(G)$ by parity.

$\mathrm{Rep}(G, z)$ - category of repr. of G on supervector spaces such that z acts by parity.

Def. A STC $\mathcal{C}$ is tannakian if $\exists$ a fiber functor (exact symmetric $\otimes$ functor) $F: \mathcal{C} \to \mathrm{Vec}_k$. In this case F is unique and we can define $G = \underline{\mathrm{Aut}}_{\otimes}(F)$ (affine group scheme $/k$) and $\mathcal{C} \cong \mathrm{Rep}\, G$.

char $k \neq 2$:

Def. A STC $\mathcal{C}$ is super-Tannakian if $\exists$ a fiber functor $F: \mathcal{C} \to s\mathrm{Vec}_k$. In this case F is unique and we can define $G = \underline{\mathrm{Aut}}_{\otimes}(F)$ (affine supergroup scheme$/k$) $z \in G(k)$ parity automorphism and $\mathcal{C} \cong \mathrm{Rep}(G, z)$.

Definition A STC $\mathcal{C}$ has

moderate growth if $\forall X \in \mathcal{C}$ there exists $C_X \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$ $\text{length}(X^{\otimes n}) \leq C_X^n$

Ex if $X \in \text{Rep}_{\Bbbk} G$, we may take $C_X = \dim_{\Bbbk} X$, so $\text{Rep}_{\Bbbk} G$ is of moderate growth.

Ex. Deligne categories $\text{Rep}\, GL_t$ are not of moderate growth.

Theorem (Deligne, 2002) A STC over $\Bbbk$ of char 0 is super-Tannakian $\Longleftrightarrow$ it is of moderate growth.

This theorem fails in characteristic p. To demonstrate it, we need to discuss the notion of semisimplification of a STC.

Given an additive rigid symmetric monoidal category $\mathcal{C}/k$ with $\mathrm{End}(\mathbb{1}) = k$, for a morphism $f : X \to X$ $(X \in \mathcal{C})$, we define its **trace** $\mathrm{Tr}(f) \in k$ as follows:

$$\underbrace{\mathbb{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^* \xrightarrow{f \otimes 1} X \otimes X^* \xrightarrow{c} X^* \otimes X \xrightarrow{\mathrm{ev}_X} \mathbb{1}}_{\mathrm{Tr}(f)}$$

In particular, the **categorical dimension** $\dim X \in k$ is $\mathrm{Tr}(1_X)$.

<u>Def.</u> A morphism $f : X \to Y$ is **negligible** if $\forall\, g : Y \to X$, $\mathrm{tr}(f \circ g) = 0$.

Lemma 1 Negligible morphisms form a tensor ideal $\mathcal{N} \subset \mathcal{C}$ (so $\mathcal{N}(X,Y) \subset \mathrm{Hom}(X,Y)$ $\forall X, Y \in \mathcal{C}$)

This means that $\mathcal{N}$ is a system of subspaces closed under composition and $\otimes$ with any morphism.

Lemma 2 Assume that the trace of any nilpotent endomorphism in $\mathcal{C}$ is 0 (e.g., $\mathcal{C}$ admits a monoidal functor into an abelian STC) Then the quotient $\overline{\mathcal{C}} = \mathcal{C}/\mathcal{N}$ (with $Ob(\overline{\mathcal{C}}) = Ob(\mathcal{C})$ and

$\mathrm{Hom}_{\overline{\mathcal{C}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y) / \mathcal{N}(X,Y)\big)$

is a semisimple STC.

Lemma 3 (D. Benson) Under this assumption, (1) if X, Y are indecomposable then $f: X \to Y$ is not negligible $\iff$ it is an isomorphism and $\dim X \neq 0$.

(2) $f = (f_{ij}) : \bigoplus_i X_i \to \bigoplus_j Y_j$ is negligible $\iff$ f_{ij} are negligible $\forall i,j$.

semisimplification

Corollary: Simple objects in $\overline{\mathcal{C}}$ are indecomposables in $\mathcal{C}$ of nonzero dimension.

$\overline{\mathcal{C}}$ = semisimplification of $\mathcal{C}$.

Example. $\mathcal{C} = \mathrm{Rep}_k(\mathbb{Z}/p)$, k of char p.

$g^p = 1 \iff (g-1)^p = 0$

Indecomposables $J_1, \ldots, J_p$ — Jordan blocks of sizes $1, \ldots, p$.

$$J_1 = \mathbb{1}$$

These define simple objects $L_1, \ldots, L_{p-1}$ of $\overline{\mathcal{C}}$

$$L_1 = \mathbb{1}$$

coming from $J_1, \ldots, J_{p-1}$.

Note that J_p gets killed since it has dimension p which is 0 in k.

Tensor product: Verlinde rule

$$L_m \otimes L_n = \bigoplus_{i=1}^{\min(m,n,p-m,p-n)} L_{|m-n|+2i-1}$$

For this reason $\overline{\mathcal{C}}$ is called the Verlinde category, denoted $\mathrm{Ver}_p(k) = \mathrm{Ver}_p$.

Variant:

Ex. $\mathrm{Ver}_2 = \mathrm{Vec}_k$

$\mathrm{Ver}_3 = s\mathrm{Vec}_k$

can replace $\mathbb{Z}/p$ with $\alpha_p = (\mathbb{G}_a)_1$

Ver_5: $X = L_3$

$X \otimes X = \mathbb{1} \oplus X$.

$\mathrm{Ver}_p^+ = \langle L_1, L_3, L_5, \dots \rangle$

$\mathrm{Ver}_p = \mathrm{Ver}_p^+ \boxtimes s\mathrm{Vec}_k$, $p>2$

So Ver_5 has no fiber functor since if F were such a functor then $\dim F(X) = d \in \mathbb{Z}_+$ would have to satisfy the equation:

$$d^2 = d + 1.$$

Another construction of Ver_p:
(Gelfand–Kazhdan, Georgiev–Mathieu, 1990s).

$\mathcal{C} = \mathrm{Tilt}(SL_2(k)) \Rightarrow$

$\bar{\mathcal{C}} = \mathrm{Ver}_p$.

So since Ver_p has no fiber functor, maybe we should consider fiber functor into Ver_p?

Theorem (V. Ostrik, 2015)

If $\mathcal{C}$ is a $\boxed{\text{fusion}}$ STC/k ($\mathrm{char}\, k = p$), i.e. finitely many simple objects + semisimple then $\exists!$ fiber functor

$F: \mathcal{C} \to \mathrm{Ver}_p$.

$G = \underline{\mathrm{Aut}}_{\otimes}(F)$

This means that

$\mathcal{C} = \mathrm{Rep}(G, \pi_1(\mathrm{Ver}_p))$ where G is a finite group scheme in Ver_p, ~ the study of such $\mathcal{C}$ thus reduces to the study of Lie theory in Ver$_p$.

What about STC which are not fusion? (moderate growth)

Simple counterexample (S. Venkatesh): char $k=2$ $\mathcal{C}=\mathrm{Rep}(k[d]/d^2)$

$\Delta d = d\otimes 1 + 1\otimes d$ $\quad H \cong k[\alpha_2]$

but $c = P\cdot R$

P - permutation,

$R = \boxed{1\otimes 1 + d\otimes d}$ ← triangular Hopf algebra.

(triangular R-matrix)

Benson - Etingof - Ostrik constructed counterexamples in char $p>2$ (they are more complicated)

So what prevents this category from having a fiber functor to $\mathrm{Ver}_2 = \mathrm{Vec}_k$?

Frobenius functor:

$\mathcal{C}$ STC in char 2.

$(1+c)^2 = 0$

$Fr(X) = \boxed{\text{Cohomology of } 1+c \text{ on } X \otimes X} = \frac{\mathrm{Ker}(1+c)}{\mathrm{Im}(1+c)}$

$\overset{\shortparallel}{X^{(1)}}$

$\Gamma^2 X \left[\begin{matrix} \Lambda^2 X \\ X^{(1)} \\ \Lambda^2 X \end{matrix} \right.$ $S^2 X$

In the above example:

$Fr(H) = 0$ (exer)

$Fr(X) = X^{(1)}$

$Fr(\mathbb{1}) = \mathbb{1}$.

SES

$$\begin{array}{c} 0 \to \mathbb{1} \to H \to \mathbb{1} \to 0 \\ \downarrow Fr \\ 0 \to \mathbb{1} \to 0 \to \mathbb{1} \to 0 \end{array}$$

$\Rightarrow$ Fr not exact on

either side!

But Fr is additive, so exact on every semisimple category, and also commutes with $\otimes$ functors. Thus

$\mathcal{C}$ does not admit a faithful tensor functor into any semisimple STC, in particular $\mathrm{Ver}_2 \; (= \mathrm{Vec}_k)$!

Frobenius functors for any p.

$X \to X^{\otimes p} \circlearrowleft c$ cyclic perm.

$c^p = 1 \implies$

So $X^{\otimes p} \in \mathcal{C} \boxtimes \mathrm{Rep}\, \mathbb{Z}/p$

We can consider its image in $\mathcal{C} \boxtimes \mathrm{Ver}_p$, get a monoidal functor (additive).

$Fr: \mathcal{C} \to \mathcal{C} \boxtimes Ver_p$. twisted-linear

$Fr(X) = \bigoplus_{i=1}^{p-1} Fr_i(X) \otimes L_i$

$Fr_i(X)$ can be described in terms of the filtration of $X^{\otimes p}$ by kernels of powers of $(1-c)^i$ (as $(1-c)^p = 0$):

$Fr_i(X) = (\mathrm{Ker}\, A \cap \mathrm{Im}\, A^{i-1}) / (\mathrm{Ker}\, A \cap \mathrm{Im}\, A^i)$, $A = 1-c$.

Def. $\mathcal{C}$ is Frobenius exact if Fr is exact.

Main Thm. (CEO 2021) sTC $\mathcal{C}$ Frobenius exact + moderate growth $\iff \exists\, F: \mathcal{C} \to Ver_p$ (and F is unique)

In particular, this holds for a semisimple STC.

Application: $\mathcal{C} = \mathrm{Rep}_k(G)$

Consider $\overline{\mathcal{C}} = \overline{\mathrm{Rep}_k(G)}$ - semisimplification

$d_n(X) = \mathrm{length}(\overline{X}^{\otimes n})$, $\overline{X}$ is the image of X in $\overline{\mathcal{C}}$.

Main th: $\exists F: \overline{\mathrm{Rep}_k(G)} \to \mathrm{Ver}_p$

$$\overline{\mathrm{Rep}_k(G)} \cong \mathrm{Rep}(\overline{G}, \pi_1)$$

$\overline{G}$ linearly reductive affine gp scheme in Ver_p

Fact

$$\sqrt[n]{\mathrm{length}(\overline{X}^{\otimes n})} \underset{\sim}{\le} \sqrt[n]{\mathrm{length}\, F(\overline{X})^{\otimes n}}$$

(same limit)

$$\Downarrow$$

$$d(Y) = \mathrm{FPdim}\, F(\overline{Y})$$

So

$$L_2 \longmapsto \begin{pmatrix} 0 & 1 & & & & 0 \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & 1 & 0 & 1 & \\ & & & 1 & 0 & 1 \\ 0 & & & & 1 & 0 \end{pmatrix}_{p=7}$$

$=$ largest eigenv. of $[Y]$, matrix by which Y acts on $\mathrm{Gr}(\mathrm{Ver}_p)$

$$F(\overline{Y}) = \sum_{i=1}^{p-1} m_i L_i$$

$$d = \sum m_i [i]_q$$

$$d(L_m) = [m]_q$$

This implies the theorem

We also get: at the beginning of the talk.

Cor. $\forall X \in \mathrm{Rep}_k(G)$ $\exists M_X \in \mathbb{R}_{>0}$ s.t. $\forall Y$ an indec. summand in $X^{\otimes n} \otimes X^{*\otimes m}$

$$d(Y) \leq M_X \quad \left(\leq (Cp)^{d(X)} \right).$$

Ex. $\mathrm{char}\, k = 2$,

$\overline{\mathrm{Rep}_k G} = \overline{\mathcal{C}} = \mathrm{Rep}_k(\overline{G})$

$\overline{G}$ – linearly reductive affine gp. scheme.

$X \in \mathrm{Rep}_k G \quad \langle X \rangle \subset \overline{\mathcal{C}}$

$\qquad\qquad\qquad\quad \| \quad$ finite type

$\qquad\qquad\quad \mathrm{Rep}_k(\overline{G}_X)$

Nagata thm.

$$1 \to A^\vee \times T \to \overline{G}_X \to \Gamma \to 1$$

$A^\vee$ dual of finite abelian 2-gp.

Γ odd order finite gp

T – torus.

Benson Conjecture: If G is a 2-group then $\forall X$ indec. of odd dim

$$X \otimes X^* = k \oplus \bigoplus \text{even dim. indec.}$$

+ order of X is ∞ or power of 2

$\Longleftrightarrow \Gamma = 1.$

$$\boxed{N(E_2)/E_2 = \Gamma}$$

In general.

This reduces to Benson conjecture for 2-groups.