Pure spinor superfield techniques in (twisted) supergravity

Fabian Hahner

Institut für Mathematik, Universität Heidelberg

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This talk is based on joint work with Ingmar Saberi (arXiv:2304.12371).

It builds on previous work with Richard Eager, Chris Elliott, Simone Noja, Johannes Walcher and Brian R. Williams.

Introduction

One interesting thing about supersymmetric field theories is the presence of protected subsectors which are sensitive to topological and holomorphic structures on spacetime.

These subsectors are extracted by *twisting*: Let $Q \in \mathfrak{g}_{\mathrm{odd}}$ with $[Q, Q] = 0$ and take invariants with respect to the odd abelian algebra spanned by Q.

The twisted theories have many desirable properties:

- They are topological-holomorphic field theories (and thus much simpler than the full theory).
- They can typically be formulated in terms of geometric moduli problems on spacetime.
- Good behavior under quantization, nice results on symmetry enhancements...

Two natural questions arise:

1. How can we compute twists efficiently?
2. What can we learn from the twists about the full theory?

I will make the case that the two problems should be addressed simultaneously.

Calculating twists is really hard. Why?

The supersymmetry transformations act in a complicated way on the fields. In particular the action is often only on-shell (there is only an L_∞ module structure).

In superspace, the supersymmetries act geometric. Twisting just means taking invariants with respect to some odd vector field.

Plan

I. How to produce universal superspace descriptions which are compatible with twisting?

$\longrightarrow$ Pure spinor superfields

II. What can we learn about the full theory from its twists?

$\longrightarrow$ Today: Eleven-dimensional supergravity

I. Pure spinor superfields and twisting

The pure spinor superfield formalism provides universal superfield descriptions of multiplets.

Let $\mathfrak{p}$ be a super Lie algebra equipped with a $\mathbb{Z}$-grading

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{t}_1[-1] \oplus \mathfrak{t}_2[-2]\,.$$

Super Poincaré algebras: $\mathfrak{p}_0 = \mathfrak{so}(d) \times \mathfrak{r}$, $\mathfrak{t}_1 = S \otimes U$ and $\mathfrak{t}_2 = V$.

We call $\mathfrak{t} = \mathfrak{p}_{>0}$ the supertranslation algebra and denote the associated super Lie group by T. ($T \sim$ superspace)

There are two actions on the free superfield $C^\infty(T)$:

$$L, R : \mathfrak{p} \longrightarrow \mathrm{Vect}(T)$$

Think of the superspace T as a supermanifold equipped with a distribution.

The nilpotence variety

$$Y = \{Q \in \mathfrak{t}_1 | [Q, Q] = 0\}$$

is the moduli space of twists for theories with $\mathfrak{p}$-symmetry.

Choose a basis Q_α of $\mathfrak{t}_1$ and e_μ of $\mathfrak{t}_2$ such that

$$[Q_\alpha, Q_\beta] = f^{\mu}_{\alpha\beta} e_\mu \,.$$

Let $R = \mathrm{Sym}^\bullet(\mathfrak{t}_1^\vee) = \mathbb{C}[\lambda^\alpha]$. The equation $[Q, Q] = 0$ defines an ideal $I = (\lambda^\alpha f^{\mu}_{\alpha\beta} \lambda^\beta)$ inside R.

The pure spinor superfield formalism is a functor

$$A^\bullet_{R/I} : \mathsf{Mod}^{\mathfrak{p}_0}_{R/I} \longrightarrow \mathsf{Mult}_{\mathfrak{p}}$$

(E, D, ρ)
field content
differential
module structure

$$A^\bullet_{R/I}(\Gamma) = (C^\infty(T) \otimes \Gamma\, ,\, \mathcal{D} = \lambda^\alpha R(Q_\alpha)\, ,\, L)$$

where the differential is induced from the right action and the $\mathfrak{p}$-module structure by the left action.

In coordinates

$$\mathcal{D} = \lambda^\alpha \left(\frac{\partial}{\partial \theta^\alpha} - f^\mu_{\alpha\beta} \theta^\beta \frac{\partial}{\partial x^\mu} \right) .$$

For each super Lie algebra $\mathfrak{p}$, there is a multiplet $A^\bullet(\mathcal{O}_Y)$ associated to the ring of functions on Y: *the canonical multiplet.*

$4d$ $\mathcal{N} = 1$: vector (BRST)

$6d$ $(2,0)$: tensor (BV)

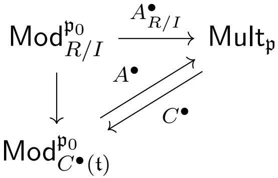
$10d$ $\mathcal{N} = 1$: SYM (BV)

$11d$ $\mathcal{N} = 1$: SUGRA (BV)

There is a derived generalization of the formalism making it an equivalence of categories.

Note that $H^0(\mathfrak{t}) = R/I \longrightarrow$ Replace R/I with $C^\bullet(\mathfrak{t})$.

The derived generalization fits into the diagram.



Here, the inverse functor is taking derived $\mathfrak{t}$-invariants:
$C^\bullet = C^\bullet(\mathfrak{t}, -)$.

The pure spinor construction is compatible with twisting.

Choose $Q \in Y$. We can twist the canonical multiplet

$$A^\bullet(\mathcal{O}_Y)^Q = (C^\infty(T) \otimes \Gamma, \mathcal{D} + Q)$$

On the other hand, we can twist the input data for the formalism.

$$\mathfrak{p} \longrightarrow \mathfrak{p}_Q = H^\bullet(\mathfrak{p}, [Q, -])$$

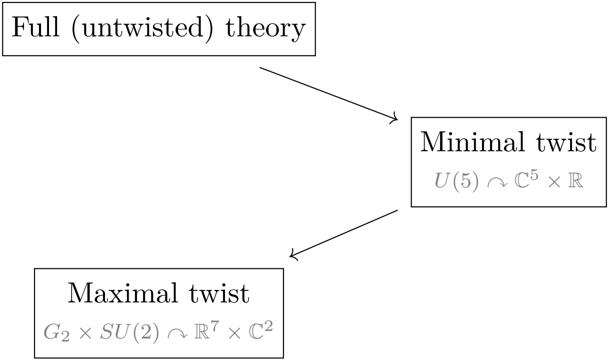
$\mathfrak{p}_Q$ is the residual symmetry algebra and has a new nilpotence variety Y_Q controlling further twists.

Both procedures are compatible [Saberi–Williams]:

$$A^\bullet(\mathcal{O}_Y)^Q \cong A^\bullet(\mathcal{O}_{Y_Q})$$

II. (Un)twisting eleven-dimensional supergravity

Eleven-dimensional supergravity has two distinct twists.



The maximal twist is Poisson–Chern–Simons theory.

$$\left(\Omega^{0,\bullet}(\mathbb{C}^2) \otimes \Omega^{\bullet}(\mathbb{R}^7)\ ,\ \bar{\partial}_{\mathbb{C}^2} + \mathrm{d}_{\mathbb{R}^7}\ ,\ \{-,-\}_{PB}\right)$$

Poisson–Chern–Simons theory

Let X be a complex manifold. We can think of the complex structure as an involutive distribution

$$T^{(0,1)}X \subset T_{\mathbb{C}}X.$$

This filtration defines a weight grading on the differential forms:

$$\left(\Omega^{\bullet}(X)\,,\,\mathrm{d}_{\mathrm{dR}} = \mathrm{d}_0 + \mathrm{d}_{-1} = \bar{\partial} + \partial\right).$$

Let X be now a Calabi–Yau 2-fold with holomorphic volume form Ω. Let $\pi = \Omega^{-1}$ be the corresponding Poisson bivector.

$$\pi : \left(\Omega^{2,\bullet}(X), \bar{\partial}\right) \longrightarrow \left(\Omega^{0,\bullet}(X), \bar{\partial}\right), \qquad \alpha \mapsto \pi \vee \alpha$$

In this situation, we can construct the Poisson bracket in the following way:

1. Turn $\Omega^\bullet(X)$ into a BV algebra.
   $\Delta = [\pi, \partial]$ and $\{\alpha, \beta\} = (-1)^{|\alpha|}(\Delta(\alpha\beta) - \Delta(\alpha)\beta) - \alpha\Delta(\beta)$
2. Define the Poisson bracket on $\Omega^{0,\bullet}(X)$ as a derived bracket.
   Set $[-,-]_\partial = \{\partial(-), -\}$ and restrict to $\Omega^{0,\bullet}$. One finds $[\alpha, \beta]_\partial = \pi(\partial\alpha \wedge \partial\beta)$.

We obtain a dg Lie algebra

$$(\Omega^{0,\bullet}(X)\,,\ \bar{\partial}\,,\ [-,-]_\partial).$$

After tensoring with the de Rham complex on an odd dimensional real manifold this gives a $\mathbb{Z}_2$-graded BV theory.

We can think about Poisson–Chern–Simons theory as based on an involutive distribution.

Almost complex structures

What if the subbundle $T^{(0,1)}X \subset T_{\mathbb{C}}X$ is not involutive?

We still get a weight grading on differential forms, but more terms appear for the differential.

$$(\Omega^\bullet(X)\,,\, \mathrm{d}_{\mathrm{dR}} = \mathrm{d}_1 + \mathrm{d}_0 + \mathrm{d}_{-1} + \mathrm{d}_{-2})$$

where

$$\mathrm{d}_1 = \bar{\mu}, \quad \mathrm{d}_0 = \bar{\partial}, \quad \mathrm{d}_{-1} = \partial, \quad \mathrm{d}_{-2} = \mu$$

and μ, $\bar{\mu}$ are coming from the Nijenhuis tensor.

We can't apply the above construction directly! Define

$$W^\bullet = H^\bullet(\Omega^\bullet(X), \mathrm{d}_1).$$

This is the appropriate generalization of the Dolbeault complex [Cirici-Wilson].

Back to superspace

In eleven dimensions, we have three different superspaces:

$$\mathfrak{p} \rightsquigarrow T, \qquad \mathfrak{p}_{Q_{\min}} \rightsquigarrow T_{Q_{\min}}, \qquad \mathfrak{p}_{Q_{\max}} \rightsquigarrow T_{Q_{\max}}$$

All three are equipped with distributions spanned by odd left invariant vector fields. Except for the maximal twist, these are non-involutive.

Consider differential forms

$$(\Omega^\bullet(T)\,,\, \mathrm{d}_{\mathrm{dR}}) \sim \mathbb{C}[x, \theta, \mathrm{d}\theta, \mathrm{d}x]$$

In a left-invariant basis $\lambda = \mathrm{d}\theta$ and $v = \mathrm{d}x + \lambda\theta$:

$$\mathrm{d}_{\mathrm{dR}} = \underbrace{\lambda^2 \frac{\partial}{\partial v}}_{\mathrm{d}_1 = \mathrm{d}_{CE}} + \underbrace{\lambda \left(\frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial x} \right)}_{\mathrm{d}_0 = \mathcal{D}} + \underbrace{v \frac{\partial}{\partial x}}_{\mathrm{d}_{-1}}$$

Let's link this to pure spinor superfields. We have

$$\Omega^\bullet(T) \cong A^\bullet(C^\bullet(\mathfrak{t})).$$

Take cohomology with respect to d_1 and perform homotopy transfer

$$h \circlearrowright (\Omega^\bullet\,,\ \mathrm{d}_1) \underset{i}{\overset{p}{\rightleftarrows}} (W^\bullet,\ 0)$$

and obtain

$$W^\bullet = (A^\bullet(H^\bullet(\mathfrak{t})), \mathrm{d}')$$

Note that

$$W^{0,\bullet} = A^\bullet(H^0(\mathfrak{t})) = A^\bullet(\mathcal{O}_Y)$$

is the canonical multiplet associated to $\mathfrak{t}$.

By computing Lie algebra cohomologies one finds that in all three cases $W^{k,\bullet}$ is concentrated in degrees $0, 1, 2$ and that there is a pairing

$$\pi : W^{2,\bullet} \longrightarrow W^{0,\bullet}.$$

$(W^{\bullet}, \mathrm{d}', \cdot, \pi)$ *has a structure similar to Dolbeault forms on an almost complex 2-fold with the multiplet* $A^{\bullet}(\mathcal{O}_Y) = W^{0,\bullet}$ *plays the role of the structure sheaf.*

$\longrightarrow$ Generalize the construction of Poisson–Chern–Simons theory to this setting.

Homotopy Poisson–Chern–Simons theory

There are appropriate generalizations of the construction of the Poisson bracket:

1. Turn $W^\bullet$ into a BV_∞ algebra.
2. Define derived brackets on $W^{0,\bullet}$.

Result: we obtain an L_∞ structure $(W^{0,\bullet}, \mu_1, \mu_2, \mu_3)$ with

$$\begin{aligned} \mu_1 &= \mathrm{d}'_0 \\ \mu_2(\alpha, \beta) &= \pi(\mathrm{d}'_{-1}\alpha \cdot \mathrm{d}'_{-1}\beta) \\ \mu_3(\alpha, \beta, \gamma) &= \pi(\mathrm{d}'_{-2}\alpha \cdot \pi(\mathrm{d}'_{-1}\beta \cdot \mathrm{d}'_{-1}\gamma)). \end{aligned}$$

We can think of homotopy Poisson–Chern–Simons theory as based on a not necessarily involutive distribution.

Applying this procedure to our three superspaces in eleven dimensions constructs:

- Poisson–Chern–Simons theory on $\mathbb{R}^7 \times \mathbb{C}^2$ from $T_{Q_{\max}}$.
- A quartic action functional for the minimal twist on $\mathbb{C}^5 \times \mathbb{R}$ from $T_{Q_{\min}}$.
- Cederwall's quartic action of eleven-dimensional supergravity in the pure spinor formalism from T.

The full theory and their twists have the same structure. We can think of eleven-dimensional supergravity as a geometric theory describing deformations of superspace equipped with a distribution.

Thank you!